B

Lebesgue Measure and Integral

In this text, two basic types of measures make an appearance. The first is Lebesgue measure on $\mathbb{R}$ (and sometimes on $\mathbb{R}^2$ or $\mathbb{R}^d$), which is used throughout the text. The second is the class of signed or complex Borel measures on $\mathbb{R}$, which mostly is restricted to Chapter **??**. Although Lebesgue measure is a special case of an unbounded Borel measure, its role in this volume is so fundamental that we choose to review the theory of Lebesgue measure and Lebesgue integration in this appendix, and separately review general Borel measures in Appendix D.

A recommended reference for Lebesgue measure and integration is the text by Wheeden and Zygmund [WZ77].

B.1 Lebesgue Measure

We begin with the familiar notion of the volume of a rectangular box in $\mathbb{R}^d$, which for simplicity we refer to as a "cube."

Definition B.1. A *cube* in $\mathbb{R}^d$ is a set of the form

$$Q \;=\; [a_1, b_1] \times \cdots \times [a_d, b_d] \;=\; \prod_{i=1}^{d} [a_i, b_i].$$

The *volume* of this cube is

$$\operatorname{vol}(Q) \;=\; (b_1 - a_1) \cdots (b_d - a_d) \;=\; \prod_{i=1}^{d} (b_i - a_i).$$

We extend the notion of volume to arbitrary sets by covering them with countably many cubes in all possible ways.

Definition B.2. The *exterior Lebesgue measure* or *outer Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e = \inf\Bigl\{\sum_k \mathrm{vol}(Q_k)\Bigr\}$$

where the infimum is taken over all all *finite or countable* collections of cubes Q_k such that $E \subseteq \bigcup_k Q_k$.

Thus, every subset of $\mathbb{R}^d$ has a uniquely defined exterior measure, which lies in the range $0 \le |E|_e \le \infty$.

It is not trivial, but it is true that the exterior measure of a cube is its volume.

Theorem B.3. (a) *If Q is a cube in $\mathbb{R}^d$ then $|Q|_e = \mathrm{vol}(Q)$.*
(b) *If $Q_1, \dots, Q_n$ are disjoint cubes in $\mathbb{R}^d$, then*

$$\Bigl|\bigcup_{k=1}^n Q_k\Bigr|_e = \sum_{k=1}^n \mathrm{vol}(Q_k).$$

Some basic properties of exterior measure are given in the next exercise.

Exercise B.4. (a) Monotonicity: If $E \subseteq F \subseteq \mathbb{R}^d$, then $|E|_e \le |F|_e$.

(b) Subadditivity: If $E_k \subseteq \mathbb{R}^d$ for $k \in \mathbb{N}$, then $\displaystyle\Bigl|\bigcup_{k=1}^\infty E_k\Bigr|_e \le \sum_{k=1}^\infty |E_k|_e$.

The next result states that every set E can be surrounded by an open set U whose exterior measure is only ε larger than that of E (by monotonicity we of course have $|E|_e \le |U|_e$).

Theorem B.5. *If $E \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, then there exists an open set $U \supseteq E$ such that*

$$|E|_e \le |U|_e \le |E|_e + \varepsilon,$$

and hence

$$|E|_e = \inf\bigl\{|U|_e : U \text{ open, } U \supseteq E\bigr\}. \tag{B.1}$$

Proof. If $|E|_e = \infty$, take $U = \mathbb{R}^d$. Otherwise, if $|E|_e < \infty$, then by definition of exterior measures, there exist cubes Q_k such that $E \subseteq \cup Q_k$ and $\sum \mathrm{vol}(Q_k) < |E| + \frac{\varepsilon}{2}$. Let Q_k^* be a larger cube that contains Q_k in its interior, and such that $\mathrm{vol}(Q_k^*) \le \mathrm{vol}(Q_k) + 2^{-k-1}\varepsilon$. Let U be the union of the interiors of the cubes Q_k^*. Then $E \subseteq U$, U is open, and

$$|U|_e \le \sum_k \mathrm{vol}(Q_k^*) \le \sum_k \mathrm{vol}(Q_k) + \frac{\varepsilon}{2} < |E| + \varepsilon. \quad \square$$

Since E and $U \backslash E$ are disjoint and their union is U, we might expect that the sum of their exterior measures is the exterior measure of U. Unfortunately, this is false in general (although the Axiom of Choice is required to construct a counterexample). Consequently, the fact that $|U|_e \le |E|_e + \varepsilon$ does *not* imply that $|U \backslash E|_e \le \varepsilon$! The "well-behaved" sets for which this is true are said to be measurable.

Definition B.6. A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable*, or simply *measurable*, if

$$\forall\, \varepsilon > 0, \quad \exists \text{ open } U \supseteq E \text{ such that } |U \backslash E|_e \le \varepsilon.$$

If E is Lebesgue measurable, then its *Lebesgue measure* is $|E| = |E|_e$. We set

$$\mathcal{L} \;=\; \{E \subseteq \mathbb{R}^d : E \text{ is Lebesgue measurable}\}.$$

Exercise B.7. Show that if $|E|_e = 0$, then $E \in \mathcal{L}$.

Consequently, if $|E|_e = 0$, then not only is E measurable, but every subset of E is measurable. In the language of abstract measure theory, the measure space $(\mathbb{R}, \mathcal{L}, |\cdot|)$ is said to be *complete*.

The following results summarize the properties of measurable sets.

Theorem B.8. (a) *$\mathcal{L}$ is a σ-algebra of subsets of $\mathbb{R}^d$. That is:*

i. *$\emptyset$, $\mathbb{R}^d \in \mathcal{L}$.*
ii. *If $E_1, E_2, \ldots \in \mathcal{L}$, then $\cup E_k \in \mathcal{L}$.*
iii. *If $E \in \mathcal{L}$, then $\mathbb{R}^d \backslash E \in \mathcal{L}$.*

(b) *Every open and every closed set belongs to $\mathcal{L}$.*

Note that since $\mathcal{L}$ is closed under complements and countable unions, it follows that it is closed under countable intersections as well, i.e., if $E_1, E_2, \dots \in \mathcal{L}$, then $\bigcap_k E_k \in \mathcal{L}$.

Theorem B.9. *Let E and E_k for $k \in \mathbb{N}$ be measurable subsets of $\mathbb{R}^d$.*

(a) Subadditivity: $\bigl|\cup E_k\bigr| \le \sum |E_k|$.
(b) Additivity: *If $E_1, E_2, \ldots$ are disjoint, then $\bigl|\cup E_k\bigr| = \sum |E_k|$.*
(c) *If $E_1 \subseteq E_2$ and $|E_2| < \infty$, then $|E_1 \backslash E_2| = |E_1| - |E_2|$.*
(d) Continuity from above: *If $E_1 \subseteq E_2 \subseteq \cdots$, then $\bigl|\cup E_k\bigr| = \lim_{k\to\infty} |E_k|$.*
(e) Continuity from below: *If $E_1 \supseteq E_2 \supseteq \cdots$ and $|E_1| < \infty$, then $\bigl|\cap E_k\bigr| = \lim_{k\to\infty} |E_k|$.*
(f) Translation Invariance: *If $h \in \mathbb{R}^d$, then $|E + h| = |E|$, where $E + h = \{x + h : x \in E\}$.*
(g) *If $T \colon \mathbb{R}^d \to \mathbb{R}^d$ is linear, then $|T(E)| = |\det(T)|\,|E|$.*

Next we give some equivalent formulations of measurability.

Definition B.10. The *inner Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_i \;=\; \sup\bigl\{|F| : F \text{ closed}, F \subseteq E\bigr\}.$$

Compare inner measure to the equivalent form of exterior measure given in equation (B.1).

Theorem B.11. *Given $E \subseteq \mathbb{R}^d$, the following statements are equivalent.*

(a) *E is Lebesgue measurable.*

(b) Carathéodory's criterion: $\forall\, A \subseteq \mathbb{R}^d,\ |A|_e \;=\; |A \cap E|_e + |A \backslash E|_e$.

If $|E|_e < \infty$, then these are also equivelent to the following statement.

(c) $|E|_e = |E|_i$.

Note that while our original definition of Lebesgue measurability, Definition B.6, was formulated in terms of the existence of surrounding open sets, the Carathéodory criterion does not depend on any topological notions. As such, the Carathéodory criterion is the appropriate definition to use to generalize measurability to abstract settings.

We end this section with some terminology. A property which holds except possibly on a set of exterior measure zero is said to hold *almost everywhere*, abbreviated a.e. For example, Problem B.1 below shows that $|C|_e = 0$, where C is the Cantor middle-thirds set, and hence the characteristic function of C is zero a.e. In other words, $\chi_C = 0$ a.e.

Following is an important example of a property that holds a.e.

Definition B.12 (Essential Supremum). The *essential supremum* of a function $f\colon E \to [-\infty, \infty]$ is

$$\operatorname*{ess\,sup}_{x \in E} f(x) \;=\; \inf\{M : f(x) \le M \text{ a.e.}\}.$$

Thus, if $M = \operatorname{ess\,sup}_{x \in E} f(x)$, then $f(x) \le M$ a.e. In particular, if $\operatorname{ess\,sup}_{x \in E} |f(x)| = 0$, then $f = 0$ a.e.

Additional Problems

B.1. Show that if $E \subseteq \mathbb{R}^d$ is countable, then $|E|_e = 0$. Show that the Cantor middle-thirds set C is an uncountable subset of $[0,1]$ which satisfies $|C|_e = 0$.

B.2. Show that if $E \subseteq \mathbb{R}^d$ and $|E|_e = 0$, then E is measurable.

Definition B.13. A subset $H \subseteq \mathbb{R}^d$ is a G_δ*-set* if there exist finitely or countably many open sets U_k such that $H = \cap U_k$.

A subset $H \subseteq \mathbb{R}^d$ is an F_σ*-set* if there exist finitely or countably many closed sets F_k such that $H = \cup F_k$.

B.3. Show that if $E \subseteq \mathbb{R}^d$, then there exists a G_δ-set $H \supseteq E$ such that $|E|_e = |H|$.

B.4. Let $E \subseteq \mathbb{R}^d$ be given. Show that the following statements are equivalent.

(a) E is Lebesgue measurable.

(b) For every $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $|E \backslash F|_e \le \varepsilon$.

(c) $E = H \backslash Z$ where H is a G_δ-set and $|Z| = 0$.

(d) $E = H \cup Z$ where H is an F_σ-set and $|Z| = 0$.

B.2 The Lebesgue Integral

There are many equivalent definitions of the Lebesgue integral. We choose to use the one that quantifies the idea that the integral of a nonnegative function should be the area under the graph of the function. In Appendix D we review the theory of abstract measures on $\mathbb{R}$, and there our definition of the integral will be based on the idea of approximating by step functions.

Definition B.14 (Region under the Graph). Let $E \subseteq \mathbb{R}^d$, and let f be an extended real-valued function on E, i.e., $f\colon E \to [0,\infty]$.

(a) The *graph* of f is the set

$$\Gamma(f,E) \;=\; \bigl\{(x,f(x)) \in \mathbb{R}^{d+1} : x \in E,\ f(x) < \infty\bigr\}.$$

(b) The *region under the graph of* f is the set $R(f,E)$ of all points $(x,y) \in \mathbb{R}^{d+1}$ with $x \in E$ and $0 \le y \le f(x)$ if $f(x) < \infty$, or $x \in E$ and $0 \le y < \infty$ if $f(x) = \infty$.

The functions that we will integrate must be measurable in the following sense.

Definition B.15 (Measurable Functions). Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable, and let $f\colon E \to [-\infty,\infty]$ be given. Then f is a *Lebesgue measurable function*, or simply a *measurable function*, if $f^{-1}(\alpha,\infty) = \{x \in \mathbb{R}^d : f(x) > \alpha\}$ is a measurable subset of $\mathbb{R}^d$ for each $\alpha \in \mathbb{R}$.

We will always implicitly assume that the domain E of a measurable function is a measurable set.

Lemma B.16. *If $E \subseteq \mathbb{R}^d$ is measurable, then $f\colon E \to [0,\infty]$ is measurable if and only if $R(f,E)$ is a measurable subset of $\mathbb{R}^{d+1}$.*

All continuous functions are measurable, but a measurable function need not be continuous.

Exercise B.17. (a) Show that if $f\colon \mathbb{R} \to [0,\infty)$ is continuous then f is measurable.

(b) Show that if $|E| = 0$, then $f = \chi_E$ is a discontinuous function that is measurable.

Lemma B.18. (a) *If $f,\ g\colon E \to [-\infty,\infty]$ are Lebesgue measurable, then so are $f+g$ and fg.*

(b) *If $f_n\colon E \to [-\infty,\infty]$ are Lebesgue measurable for $n \in \mathbb{N}$, then so are $\sup f_n$, $\inf f_n$, $\limsup f_n$, and $\liminf f_n$. Consequently, if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each x, then f is measurable.*

Definition B.19 (Lebesgue Integral of a Nonnegative Function). If $f\colon \mathbb{R} \to [0,\infty]$ is a measurable function, then the *Lebesgue integral of f over E* is the measure of the region under the graph of f as a subset of $\mathbb{R}^{d+1}$, denoted

$$\int_E f \;=\; \int_E f(x)\,dx \;=\; |R(f,E)|.$$

If $E = \mathbb{R}^d$, then we write simply $\int f$ or $\int f(x)\,dx$ for the integral over $\mathbb{R}^d$.

Following are some basic properties of integrals of nonnegative functions.

Theorem B.20. *Let $E \subseteq \mathbb{R}^d$ be measurable, and let $f,\ g\colon E \to [0,\infty]$ be measurable functions. Then the following statements hold.*

(a) $\int_E 1 = |E|$.

(b) *If $f \le g$, then $\int_E f \le \int_E g$.*

(c) *If $E_1 \subseteq E_2$ are measurable, then $\int_{E_1} f \le \int_{E_2} f$.*

(d) *If $E_1, E_2, \dots$ are disjoint measurable sets and $E = \cup E_k$, then $\int_E f = \sum_k \int_{E_k} f$.*

(e) $\int_E (f+g) = \int_E f + \int_E g$.

(f) Tchebyshev's Inequality: *If $\alpha > 0$, then $|\{x \in E : f(x) > \alpha\}| \le \frac{1}{\alpha}\int_E f$.*

(g) *$f = 0$ a.e. on E if and only if $\int_E f = 0$.*

(h) *If $f = g$ a.e., then $\int_E f = \int_E g$.*

Exercise B.21. Parts (b), (c), (d), (f), (g), and (h) of Theorem B.20 follow directly from the definition of the integral and previous results. Prove these parts.

On the other hand, parts (a) and (e) of Theorem B.20 are seemingly "obvious" properties whose proofs are actually rather technical, given our choice of definition of the integral (compare Exercise D.33 for abstract measures).

We define the integral of a general real-valued function by writing it as a difference of two nonnegative functions, and a complex-valued function by breaking into the real and imaginary parts.

Definition B.22 (Lebesgue Integral of a Real-Valued Function). Let $E \subseteq \mathbb{R}^d$ and a measurable extended real-valued function $f\colon E \to [-\infty,\infty]$ be given. Define

$$f^+(x) \;=\; \max\bigl\{f(x),\,0\bigr\}, \qquad f^-(x) \;=\; \max\bigl\{-f(x),\,0\bigr\},$$

and note that that

$$f \;=\; f^+ - f^-, \qquad |f| \;=\; f^+ + f^-.$$

We define the *Lebesgue integral of f on E* to be

$$\int_E f \;=\; \int_E f^+ \;-\; \int_E f^-,$$

as long as this does not have the form $\infty - \infty$ (in that case, the integral is undefined).

Definition B.23 (Lebesgue Integral of a Complex-Valued Function). Let $E \subseteq \mathbb{R}^d$ and a complex-valued function $f\colon E \to \mathbb{C}$ be given. We say that f is *measurable* if both $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are measurable. If $\int_E \operatorname{Re}(f)$ and $\int_E \operatorname{Im}(f)$ both exist, then the *Lebesgue integral of f on E* is

$$\int_E f \;=\; \int_E \operatorname{Re}(f) \;+\; i\int_E \operatorname{Im}(f).$$

Exercise B.24. Given an extended real-valued or a complex-valued measurable function f, show that $\int_E f$ exists and is a finite scalar if and only if $\int_E |f| < \infty$.

Additional Problems

B.5. Given $E \subseteq \mathbb{R}^d$, show that E is a measurable set if and only if χ_E is a measurable function.

B.6. Show that $f\colon \mathbb{R}^d \to \mathbb{R}$ is measurable if and only if $f^{-1}(U)$ is measurable for every open $U \subseteq \mathbb{R}$.

B.7. Show that if $f\colon \mathbb{R}^d \to \mathbb{R}$ is measurable and $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is measurable. Conclude that $|f|$, f^2, f^+, f^-, $|f|^p$ for $p > 0$, etc. are measurable.

B.3 The L^p Spaces

In this section we introduce an important class of Banach spaces.

Definition B.25 (Integrable Function). Let $E \subseteq \mathbb{R}^d$ be measurable. Then a measurable function f on E (either extended real-valued or complex-valued) is *integrable* on E if $\int_E |f| < \infty$.

Definition B.26. Let $E \subseteq \mathbb{R}^d$ be measurable.

(a) If $0 < p < \infty$, then $L^p(E)$ consists of all measurable functions $f\colon E \to \mathbb{C}$ such that $|f|^p$ is integrable, i.e.,

$$\|f\|_p \;=\; \left(\int_E |f|^p\right)^{1/p} \;<\; \infty.$$

(b) For $p = \infty$, the space $L^\infty(E)$ consists of all measurable functions $f\colon E \to \mathbb{C}$ such that $|f|$ is essentially bounded, i.e.,

$$\|f\|_\infty \;=\; \operatorname*{ess\,sup}_{x\in E} |f(x)| \;<\; \infty.$$

Hölder's Inequality, given in Theorem A.18 for the ℓ^p spaces, also holds for $L^p(E)$.

Theorem B.27 (Hölder's Inequality). *Let $E \subseteq \mathbb{R}$ be measurable. Given $1 \le p \le \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$, if $f \in L^p(E)$ and $g \in L^{p'}(E)$, then then $fg \in L^1(E)$, and*

$$\|fg\|_1 \le \|f\|_p \, \|g\|_{p'}.$$

For $1 < p < \infty$, this inequality is

$$\int_E fg \;\le\; \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^{p'}\right)^{1/p'}.$$

In Exercise A.19 we saw that the ℓ^p spaces are Banch spaces for $p \ge 1$. An analogous result holds for $L^p(E)$, but the fact that $\int_E |f|^p = 0$ only implies that $f = 0$ a.e. adds a technical complication. The triangle inequality on L^p is also called *Minkowski's Inequality.*

Theorem B.28. *If $E \subseteq \mathbb{R}^d$ be measurable and $1 \le p \le \infty$, then $\|\cdot\|_p$ is a seminorm on $L^p(E)$.*

The next exercise is the standard procedure for converting a seminorm to a norm by forming equivalence classes.

Exercise B.29. Show that the relation $f \sim g$ if $f = g$ a.e. is an equivalence relation on $L^p(E)$. Let $\tilde{f}$ denote the equivalence class of f in $L^p(E)$ under this relation, and set $|||\tilde{f}|||_p = \|f\|_p$. Show that this quantity is independent of the choice of representative f. Let the quotient space $\widetilde{L^p}(E)$ consist of all the distinct equivalence classes of $f \in L^p(E)$, and show that $\widetilde{L^p}(E)$ is a normed space with respect to $|||\cdot|||_p$.

Typically we abuse notation and let the symbol f denote the equivalence class $\tilde{f}$ of all functions equal to f a.e., and we write $L^p(E)$ instead of $\widetilde{L^p}(E)$. In other words, we identify any two functions that are equal a.e. Ignoring the distinction between a function and the equivalence class of functions that are equal to it a.e. is not usually a problem, but on occasion care should be taken, as when dealing with a continuous function.

Example B.30. Every function in $C_b(\mathbb{R})$ is continuous and bounded. Therefore we write $C_b(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, but in doing so we are really identifying $C_b(\mathbb{R})$ with its image in $L^\infty(\mathbb{R})$ under the equivalence relation $\sim$, i.e., if $f \in C_b(\mathbb{R})$ then it determines an equivalence class of functions in $L^\infty(\mathbb{R})$ that are equal to it almost everywhere. Conversely, if we are given $f \in L^\infty(\mathbb{R})$ (really an equivalence class of functions), then the statement $f \in C_b(\mathbb{R})$ means that there is a representative of this equivalence class that belongs to $C_b(\mathbb{R})$.

Exercise B.31. Show that if $f \in C_b(\mathbb{R})$, then

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| \;=\; \sup_{x\in\mathbb{R}} |f(x)|.$$

Consequently, for continuous bounded functions, the uniform norm defined in Definition A.21 coincides with the L^∞-norm defined above.

With these identifications, and similar ones for $L^p(E)$ when $p < 1$, we have the following result.

Theorem B.32. *Let $E \subseteq \mathbb{R}^d$ be measurable.*

(a) *If $1 \le p \le \infty$, then $\|\cdot\|_p$ is a norm on $L^p(E)$, and $L^p(E)$ is a Banach space with respect to this norm.*

(b) *If $0 < p < 1$, then $\mathrm{d}(f,g) = \|f-g\|_p^p$ is a metric on $L^p(E)$, and $L^p(E)$ is complete with respect to this metric.*

In the ℓ^p spaces, convergence in ℓ^p norm implies componentwise convergence (see Problem A.4). The situation in $L^p(E)$ is a little different — an L^p-convergent sequence need not converge pointwise.

Exercise B.33. Let $1 \le p < \infty$ be fixed. Give an example of continuous functions $f_n \in L^p(\mathbb{R})$ such that $f_n \to 0$ in L^p-norm (i.e., $\|f - f_n\|_p \to 0$), but $f_n(x)$ does not converge pointwise to zero as $n \to \infty$. Give another example in $L^p[0,1]$.

Fortunately, it is true that an L^p-convergent sequence always has a subsequence that converges pointwise almost everywhere.

Theorem B.34. *Let $E \subseteq \mathbb{R}^d$ be measurable and choose $1 \le p \le \infty$. If f_n, $f \in L^p(E)$ and $f_n \to f$ in L^p-norm, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k}(x) \to f(x)$ for a.e. $x \in E$.*

The space $L^2(E)$ is special. As before, we consider elements of $L^2(E)$ to be equivalence classes of functions that are equal almost everywhere.

Exercise B.35. If $E \subseteq \mathbb{R}^d$ is measurable, show that

$$\langle f, g\rangle \;=\; \int_E f(x)\,\overline{g(x)}\,dx$$

defines an inner product on $L^2(E)$, and $L^2(E)$ is a Hilbert space with respect to this inner product.

Additional Problems

B.8. Show that $C_0(\mathbb{R})\backslash L^1(\mathbb{R}) \ne \emptyset$, and $(L^1(\mathbb{R}) \cap C_b(\mathbb{R}))\backslash C_0(\mathbb{R}) \ne \emptyset$.

B.9. Show that if $|E| < \infty$ and $1 \le p \le q \le \infty$, then $L^q(E) \subseteq L^p(E)$.

B.10. Define $e_n(x) = e^{2\pi i n x}$. Show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal sequence in $L^2[0,1]$.

B.4 Convergence Theorems

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that converge pointwise almost everywhere to a function f, then it need not be true that $\lim_{n\to\infty} \int_E f_n$ will converge to $\int_E f$. In this section we review several important theorems related to this issue.

The following result is also known as the *Beppo-Levi Theorem*. We say that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is *monotone increasing* if $f_1(x) \le f_2(x) \le \cdots$ for all x. We write $f_n \nearrow f$ to denote that $\{f_n\}_{n\in\mathbb{N}}$ is monotone increasing and $f_n(x) \to f(x)$ pointwise.

Theorem B.36 (Monotone Convergence Theorem). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable, nonnegative, and monotone increasing functions on a measurable set $E \subseteq \mathbb{R}^d$, and define $f(x) = \lim_{n\to\infty} f_n(x)$, i.e., $f_n \nearrow f$. Then*

$$\lim_{n\to\infty} \int_E f_n \;=\; \int_E f.$$

Proof. Note that $R(f_1, E) \subseteq R(f_2, E) \subseteq \cdots$ and that $R(f, E) = \cup R(f_n, E)$. Theorem B.9(d) therefore implies that $|R(f_n, E)|$ converges to $|R(f, E)|$.

Since changing the value of a function on a set of zero measure does not change the value of its integral, it sufficies to assume that the hypotheses in the Monotone Convergence Theorem and the other theorems in this section hold a.e. instead of everywhere.

Since the partial sums of a series of nonnegative functions form a monotone increasing sequence, we obtain the following corollary regarding the interchange of a sum and an integral.

Corollary B.37. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable, nonnegative functions on a measurable set $E \subseteq \mathbb{R}^d$. Then*

$$\int_E \biggl(\sum_{n=1}^{\infty} f_n\biggr) \;=\; \sum_{n=1}^{\infty} \int_E f_n.$$

If the functions f_n are nonnegative but are not monotone increasing, then we may not be able to interchange a limit and an integral. However, the following result states that if the functions f_n are all nonnegative, then we do at least have a particular inequality.

Theorem B.38 (Fatou's Lemma). *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable, nonnegative functions on measurable set $E \subseteq \mathbb{R}^d$, then*

$$\int_E \Bigl(\liminf_{n\to\infty} f_n\Bigr) \;\le\; \liminf_{n\to\infty} \int_E f_n.$$

The following result is an extremely useful convergence theorem.

Theorem B.39 (Lebesgue Dominated Convergence Theorem). *Let $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable, nonnegative functions on measurable set $E \subseteq \mathbb{R}^d$ such that:*

(a) $f(x) = \lim_{n\to\infty} f_n(x)$ *exists for a.e.* $x \in E$, *and*

(b) *there exists* $g \in L^1(E)$ *such that* $|f_n(x)| \le g(x)$ *a.e. for every* n.

Then f_n *converges to* f *in* L^1*-norm, i.e.,*

$$\lim_{n\to\infty} \|f - f_n\|_1 \;=\; \lim_{n\to\infty} \int_E |f - f_n| \;=\; 0,$$

and, consequently,

$$\lim_{n\to\infty} \int_E f_n \;=\; \int_E f.$$

Additional Problems

B.11. Give examples showing that strict inequality can hold in Fatou's Lemma.

B.12. Prove the Monotone Convergence Theorem.

B.13. Prove Fatou's Lemma.

B.14. Prove the Lebesgue Dominated Convergence Theorem.

B.5 Repeated Integration

Let $E \subseteq \mathbb{R}^m$ and y and $F \subseteq \mathbb{R}^n$ be measurable. If f is a measurable function on $E \times F$ then there are three natural integrals of f on $E \times F$. First, there is the integral of f on $E \times F$ as a subset of $\mathbb{R}^{m+n}$, which we write as the *double integral*

$$\iint_{E\times F} f \;=\; \iint_{E\times F} f(x,y)\,(dx\,dy).$$

Second, for each fixed y we can integrate $f(x,y)$ as a function of x, and then integrate the result in y, obtaining the *iterated integral*

$$\int_F \left(\int_E |f(x,y)|\,dx \right) dy.$$

Third, we also have the iterated integral

$$\int_E \left(\int_F |f(x,y)|\,dy \right) dx.$$

It general these three integrals need not be equal, even if they all exist.

In this section we review Fubini's and Tonelli's Theorems, which give sufficient conditions under which we can exchange the order of integration. First, Tonelli's Theorem states that interchange is allowed if f is nonnegative.

Theorem B.40 (Tonelli's Theorem). *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If $f\colon E \times F \to [0,\infty]$ is measurable, then the following statements hold.*

(a) $f_x(y) = f(x,y)$ *is measurable on F for almost every $x \in E$.*

(b) $f^y(x) = f(x,y)$ *is measurable on E for almost every $y \in F$.*

(c) $g(x) = \int_F f_x(y)\,dy$ *is a measurable function on E.*

(d) $h(y) = \int_E f^y(x)\,dx$ *is a measurable function on F.*

(e) *We have*

$$\iint_{E\times F} f(x,y)\,(dx\,dy) = \int_F \left(\int_E f(x,y)\,dx\right) dy = \int_E \left(\int_F f(x,y)\,dy\right) dx.$$

As a corollary, we obtain the useful fact that to test whether a given function belongs to $L^1(E \times F)$ we can simply show that any one of three possible integrals is finite.

Corollary B.41. *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If f is a measurable function on $E \times F$, then*

$$\iint_{E\times F} |f(x,y)|\,(dx\,dy) = \int_F \left(\int_E |f(x,y)|\,dx\right) dy = \int_E \left(\int_F |f(x,y)|\,dy\right) dx.$$

Consequently, if any one of these three integrals is finite, then $f \in L^1(E \times F)$.

Fubini's Theorem allows the interchange of integrals if f is integrable.

Theorem B.42 (Fubini's Theorem). *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If $f \in L^1(E \times F)$, then the following statements hold.*

(a) $f_x(y) = f(x,y)$ *is measurable and integrable on F for almost every $x \in E$.*

(b) $f^y(x) = f(x,y)$ *is measurable and integrable on E for almost every $y \in F$.*

(c) $g(x) = \int_F f_x(y)\,dy$ *is a measurable and integrable function on E.*

(d) $h(y) = \int_E f^y(x)\,dx$ *is a measurable and integrable function on F.*

(e) *We have*

$$\iint_{E\times F} f(x,y)(dx\,dy) = \int_F \left(\int_E f(x,y)\,dx\right) dy = \int_E \left(\int_F f(x,y)\,dy\right) dx.$$

B.6 Functions of Bounded Variation