4

Integration

In this chapter we will develop the theory of integration of functions with respect to general measures.

4.1 Approximation by Simple Functions

Often, the easiest way to deal with a generic measurable function is to approximate it by simpler functions. Of course, the meaning of "simpler" is in the eye of the beholder, but one way in which a function φ can be "simple" is if it only takes finitely many different values. We will not require that the set on which φ takes a particular value have any special structure aside from being measurable. All we will require of a "simple function" is that it is measurable and takes only finitely many real or complex values (infinity is not allowed). The precise definition is as follows.

Definition 4.1. Let (X, Σ) be a measurable space. A *simple function* on X is a measurable function $\varphi\colon X \to \mathbb{C}$ that takes only finitely many distinct values. $\diamondsuit$

A simple function can be real-valued, but it *cannot* take the values $\pm\infty$. In order for φ to be called a simple function, φ must be measurable, $\varphi(x)$ must be a real or complex scalar for each $x \in X$, and the set of all values of φ must be a finite set. The set of all values is just another name for

$$\operatorname{range}(\varphi) \;=\; \bigl\{\varphi(x) : x \in X\bigr\},$$

so a simple function is a measurable function whose range is a finite subset of $\mathbb{C}$.

Example 4.2. Aside from the zero function, the simplest example of a simple function is the characteristic function of a measurable set $A \subseteq X$, which is defined explicitly as

$$\chi_A(x) \;=\; \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

A characteristic function takes only two values, and from Example 3.18 we know that χ_A is a measurable function if and only if A is a measurable subset of X. ♢

We can make more "interesting" simple functions by forming finite linear combinations of characteristic functions. Specifically, if $E_1, \dots, E_N$ are measurable subsets of X and $c_1, \dots, c_N$ are complex scalars, then the function $\varphi = \sum_{k=1}^N c_k \, \chi_{E_k}$ takes only finitely many values and is measurable by Exercise 3.45, so φ is a simple function on X. In fact, the next lemma (whose proof essentially follows "from inspection") states that every simple function has this form.

Lemma 4.3. *Let φ be a simple function on a measurable space (X, Σ). If $c_1, \dots, c_N$ are the distinct values taken by φ and we set*

$$E_k \;=\; \varphi^{-1}\{c_k\} \;=\; \{\varphi = c_k\}, \qquad k = 1, \dots, N, \tag{4.1}$$

then

$$\varphi \;=\; \sum_{k=1}^{N} c_k \, \chi_{E_k}.$$

Moreover, the sets $E_1, \dots, E_N$ defined in equation (4.1) *partition X into disjoint measurable sets.* ♢

There may be many ways to write a given simple function as a linear combination of characteristic functions, but we encounter the particular form given in Lemma 4.3 often enough that we give it a special name.

Definition 4.4. The *standard representation* of a simple function φ is the representation given by Lemma 4.3, i.e., $\varphi = \sum_{k=1}^N c_k \, \chi_{E_k}$ where $c_1, \dots, c_N$ are the distinct values taken by φ and $E_k = \{\varphi = c_k\}$. ♢

For example, the standard representation of $\varphi = \chi_{[0,2]} + \chi_{[1,3]}$ is

$$\varphi \;=\; 0\,\chi_{E_1} + 1\,\chi_{E_2} + 2\,\chi_{E_3},$$

where $E_1 = (-\infty, 0) \cup (3, \infty)$, $E_2 = [0,1) \cup (2,3]$, and $E_3 = [1,2]$. Of course, we can also write φ in the form

$$\varphi \;=\; \chi_{E_2} + 2\,\chi_{E_3},$$

but while the sets E_2, E_3 are disjoint, they do not partition $\mathbb{R}$. In general, one of the scalars c_k in the standard representation of a simple function φ might be zero.

If $\varphi = \sum_{j=1}^{M} c_j \chi_{A_j}$ and $\psi = \sum_{k=1}^{N} d_k \chi_{B_k}$ are the standard representations of the simple functions φ and ψ, then

$$\varphi + \psi \;=\; \sum_{j=1}^{M}\sum_{k=1}^{N} (c_j + d_k)\,\chi_{A_j \cap B_k}.$$

This need not be the standard representation of $\varphi+\psi$, since the scalars c_j+d_k may coincide for different values of j and k. However, it does show that the sum of two finite simple functions is finite. A similar idea applied to products establishes the next lemma.

Lemma 4.5. *The class of simple functions on a measurable space* (X, Σ) *is closed with respect to addition, scalar multiplication, and products. That is, if* φ *and* ψ *are simple functions on* X *and* $c \in \mathbb{C}$, *then*

$$\varphi + \psi, \qquad c\varphi, \qquad \varphi\,\psi,$$

are all simple functions on X. ♢

Much of the power of simple functions lies in the next theorem, which states that every nonnegative function (including those that take the value ∞) can be written as a pointwise limit of a sequence of simple functions ϕ_k. In fact, we will be able to construct the simple functions ϕ_k so that they increase pointwise to f (which we denote by writing $\phi_k \nearrow f$), and the convergence is uniform on any set where f is bounded.

Theorem 4.6. *Let* (X, Σ) *be a measurable space. If* $f \colon X \to \overline{\mathbb{R}}$ *is measurable, then there exist nonnegative simple functions* $\phi_1, \phi_2, \dots$ *that increase pointwise to* f, *i.e.,*

(a) $0 \le \phi_1 \le \phi_2 \le \cdots \le f$, *and*

(b) $\lim_{k\to\infty} \phi_k(x) = f(x)$ *for each* $x \in X$.

Moreover, if f *is bounded on some set* $E \subseteq X$, *then we can construct the functions* ϕ_k *so that they converge uniformly to* f *on* E, *i.e.,*

$$\lim_{k\to\infty} \Bigl(\sup_{x\in E} |f(x) - \phi_k(x)|\Bigr) \;=\; 0.$$

Proof. The idea of the proof is that we construct ϕ_k by simply rounding f down to the nearest integer multiple of 2^{-k}. However, if f is unbounded then this would give ϕ_k infinitely many values, while a simple function can only take finitely many values. Hence we stop the rounding down process at some finite height. Typical choices for this height are k or 2^k; we will use the former.

Thus, for $k = 1$ we define ϕ_1 by rounding f down to the nearest integer, with the caveat that we stop at height 1 (see the illustration in Figure 4.1).

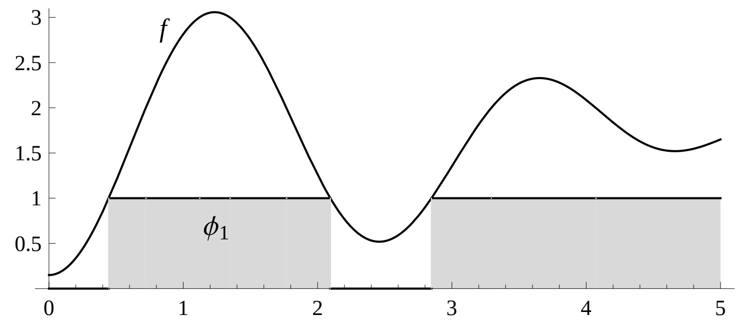

Fig. 4.1. Graphs of a function f and the approximating simple function ϕ_1 (the region under the graph of ϕ_1 is shaded).

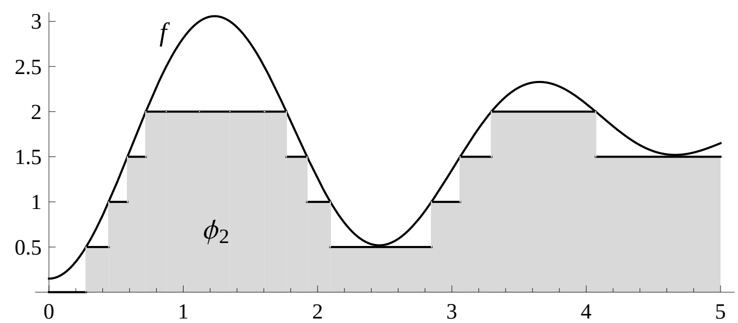

Fig. 4.2. Graphs of a function f and the approximating simple function ϕ_2 (with shading under the graph of ϕ_2).

Specifically,

$$\phi_1(x) = \begin{cases} 0, & 0 \le f(x) < 1, \\ 1, & f(x) \ge 1. \end{cases}$$

For ϕ_2 we round down to the nearest integer multiple of $\frac{1}{2}$, except we never exceed height 2 (see Figure 4.2):

$$\phi_2(x) = \begin{cases} 0, & 0 \le f(x) < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \le f(x) < 1, \\ 1, & 1 \le f(x) < \frac{3}{2}, \\ \frac{3}{2}, & \frac{3}{2} \le f(x) < 2, \\ 2, & f(x) \ge 2. \end{cases}$$

Note that if $f(x) \le 2$, then $f(x)$ and $\phi_k(x)$ differ by at most $\frac{1}{2}$.

In general, given a positive integer k we define ϕ_k by

$$\phi_k(x) = \begin{cases} \frac{j-1}{2^k}, & \frac{j-1}{2^k} \le f(x) < \frac{j}{2^k},\ j = 1, \dots, k2^k, \\ k, & f(x) \ge k, \end{cases} \tag{4.2}$$

The sets

$$\left\{\frac{j-1}{2^k} \le f < \frac{j}{2^k}\right\} \qquad \text{and} \qquad \{f \ge k\}$$

are measurable because f is measurable, so it follows that ϕ_k is a measurable function. Further, by construction we have $\phi_k(x) \le \phi_{k+1}(x)$ for every x, and

$$f(x) \le k \quad \Longrightarrow \quad |f(x) - \phi_k(x)| \le 2^{-k}.$$

If $f(x)$ is finite, then k will eventually exceed $f(x)$, so we have $\phi_k(x) \to f(x)$ in this case. In fact, if $f(x) \le M < \infty$ for all x in some set E, then

$$\sup_{x \in E} |f(x) - \phi_k(x)| \le 2^{-k} \quad \text{for } k \ge M.$$

Hence ϕ_k converges uniformly to f on E in this case. On the other hand, if $f(x) = \infty$ then $\phi_k(x) = k$ for every k, so $\phi_k(x) \to f(x)$ as $k \to \infty$, and thus we still have pointwise convergence in this case. □

Theorem 4.6 will be especially useful to us when we define integration in Chapter 4.

We can extend Theorem 4.6 to the case of an extended real-valued function f by writing $f = f^+ - f^-$ where f^+ and f^- are the positive and negative parts of f. Complex-valued functions can be treated similarly by breaking into real and imaginary parts. We assign the proof of this as Problem 4.2.

4.1.1 Lusin's Theorem

As an application of approximation by simple functions and Egorov's Theorem, we will prove Lusin's Theorem. which states that a measurable function on the interval $[a, b]$ is a continuous function on "most" of its domain. Although "Lusin" is usually spelled with an *s* when referring to this theorem, a better spelling would be "Luzin's Theorem," since it is named for the Russian mathematician Nikolai Luzin (1883–1950).

Theorem 4.7 (Lusin's Theorem). *Let $f\colon [a, b] \to \mathbb{C}$ be a Lebesgue measurable function. Given any $\varepsilon > 0$, there exists a continuous function $g\colon [a, b] \to \mathbb{C}$ such that $|\{f \ne g\}| < \varepsilon$.*

Additional Problems

4.1. Let (X, Σ) be a measurable space. Suppose that $f\colon X \to \mathbb{R}$ takes only finitely many distinct values, and let $E_1, \dots, E_N$ be the corresponding disjoint subsets of X on which these values are taken. Show that f is measurable if and only if $E_1, \dots, E_N$ are all measurable.

4.2. Let f be a measurable function, either extended real-valued or complex-valued, on a measurable space (X, Σ). Show that there exist simple functions ϕ_k such that:

(a) $\phi_k \to f$ pointwise as $k \to \infty$,

(b) $|\phi_k(x)| \leq |f(x)|$ for every k and x,

(c) the convergence is uniform on every set on which f is bounded.