3.8 Three Types of Convergence

Suppose that we are given a sequence functions $\{f_k\}_{k\in\mathbb{N}}$ on a set X and another function f on X. What does it mean for f_k to converge to f? Loosely speaking, convergence means that f_k becomes "closer and closer" to f as k increases. However, there are many ways to say exactly what we mean by "close." Two familiar ways to quantify convergence are pointwise convergence and uniform convergence. These types of convergence were discussed in Sections 0.1 and 0.2, respectively. In this section we will give an "almost everywhere" version of pointwise and uniform convergence, and then introduce a new notion that we call "convergence in measure." Each of these types of convergence (and others that we will encounter later!) have their own important role to play in analysis. We need all of these different notions in different circumstances; which one to use will depend on the application that we have at hand.

3.8.1 Pointwise Almost Everywhere Convergence

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions on a set X, either complex-valued or extended real-valued. Recalling Definition 0.5, we say that f_k converges pointwise to a function f if for each *individual* element $x \in X$, the scalar $f_k(x)$ converges to $f(x)$ as $k \to \infty$. We often write $f_k \to f$ *pointwise* to denote pointwise convergence.

We do not need to have a measure on a domain X in order to discuss pointwise convergence on X. However, if we do have a measure then it is often the case that sets of measure zero are simply not important when considering convergence. Therefore, a useful variation on pointwise convergence is *pointwise almost everywhere convergence*, which is pointwise convergence with the exception of a set of points whose measure is zero. For example, this is the type of convergence that is used in the statement of part (b) of Corollary 3.48. Here is a precise definition.

Definition 3.49 (Pointwise Almost Everywhere Convergence). Let (X, Σ, μ) be a measure space, and let f_k, f be measurable functions on E that are either extended real-valued or complex-valued. We say that f_k *converges pointwise almost everywhere to* f if there exists a measurable set $Z \subseteq X$ such that $\mu(Z) = 0$ and

$$\forall\, x \in X\backslash Z, \quad \lim_{k\to\infty} f_k(x) \;=\; f(x).$$

We often denote pointwise almost everywhere convergence by writing $f_k \to f$ *pointwise* μ*-a.e.* or simply $f_k \to f$ μ*-a.e.* (and we may also omit writing the symbol μ if it is understood). $\diamondsuit$

Often, the functions f_k not only converge pointwise a.e. to f, but are also monotone increasing except on a set of measure zero, i.e.,

$$f_1(x) \le f_2(x) \le \cdots \qquad \text{for a.e. } x.$$

In this case we say that that f_k *increases pointwise a.e. to* f, and denote this by writing $f_k \nearrow f$ μ-a.e., or just $f_k \nearrow f$ a.e. A similar definition is made for decreasing sequences.

Remark 3.50. Note that if μ is a complete measure, then Corollary 3.48 implies that measurability of the limit function f in Definition 3.49 follows automatically from the measurability of the functions f_k. $\diamondsuit$

Example 3.51. The functions $\chi_{(0,\frac{1}{k})}$ converge pointwise to the zero function as $k \to \infty$. On the other hand, the functions $\chi_{[0,\frac{1}{k}]}$ do converge pointwise to the zero function, for if $x = 0$ then the limit is 1, not zero. However, the singleton $\{0\}$ has Lebesgue measure zero, so $\chi_{[0,\frac{1}{k}]} \searrow 0$ a.e. $\diamondsuit$

3.8.2 L^∞ Convergence

Uniform convergence is a stronger requirement than pointwise convergence in that it requires a "simultaneity" of convergence over all of the domain rather just "individual" convergence at each x. As discussed in Section 0.2, a convenient way to view uniform convergence is in terms of the *uniform norm*

$$\|f\|_{\mathrm{u}} = \sup_{x \in X} |f(x)|. \tag{3.7}$$

Uniform convergence requires that the distance between f_k and f, as measured by the uniform norm, shrinks to zero as k increases. That is,

$$f_k \to f \text{ uniformly} \quad \Longleftrightarrow \quad \lim_{k\to\infty} \|f - f_k\|_{\mathrm{u}} = 0.$$

We can define uniform convergence without needing to have a measure on our domain X. However, if we do have a measure on X then sets with measure zero often "don't matter." In keeping with this philosophy, we modify the definition of uniform convergence by simply replacing the supremum that appears in equation (3.7) with an essential supremum. We introduced the essential supremum for functions on $\mathbb{R}^d$ in Definition 1.47, and the following definition extends this to functions on an arbitrary measure space.

Definition 3.52 (Essential Supremum). Let (X, Σ, μ) be a measure space. The *essential supremum* of a measurable function $f\colon X \to \overline{\mathbb{R}}$ is

$$\operatorname*{ess\,sup}_{x \in X} f(x) = \inf\bigl\{M : f(x) \le M \ \mu\text{-a.e.}\bigr\}. \qquad \diamondsuit \tag{3.8}$$

We give the following name to the essential supremum of $|f|$. This name may not seem very inspired at this point, but it will fit naturally for the terminology for the L^p spaces that we will introduce in Chapter 5.

Definition 3.53 (L^∞ norm). Let (X, Σ, μ) be a measure space, and let f be an measurable function on X, either extended real-valued or complex-valued. The L^∞ *norm* of f is the essential supremum of $|f|$, and it is denoted by

$$\|f\|_\infty \;=\; \operatorname*{ess\,sup}_{x \in X} |f(x)|. \qquad \diamondsuit$$

Problem 1.27, which was stated specifically for Lebesgue measure, generalizes to abstract measures without change. Consequently, we have

$$|f(x)| \;\le\; \|f\|_\infty \quad \text{for a.e. } x \in X.$$

Further, by the definition of supremum we have the inequality

$$\|f\|_\infty \;\le\; \|f\|_{\mathrm{u}}.$$

However, the uniform norm and the L^∞ norm of f need not be equal. For example, if we take $X = \mathbb{R}$ and define $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$ then we have $\|f\|_\infty = 0$ while $\|f\|_{\mathrm{u}} = 1$. On the other hand, Problem 1.29 showed that if f is a continuous function on $\mathbb{R}^d$ then $\|f\|_\infty = \|f\|_{\mathrm{u}}$. Hence in some ways we can regard the L^∞ norm as being a generalization of the uniform norm.

We introduce some additional terminology associated with the essential supremum and the L^∞ norm.

Definition 3.54 (Essentially Bounded Function). Let (X, Σ, μ) be a measure space, and let f, f_k, g be measurable functions on X, either extended real-valued or complex-valued.

(a) We say that f is *essentially bounded* if $\|f\|_\infty < \infty$.

(b) The quantity $\|f - g\|_{\mathrm{u}}$ is called the L^∞ *distance between f and g*.

(c) We say that f_k *converges to f in L^∞ norm* if

$$\lim_{k \to \infty} \|f - f_k\|_\infty \;=\; 0. \qquad \diamondsuit$$

The next exercise shows that the L^∞ norm *almost* satisfies the properties given in Definition 0.1 that are necessary in order for $\|\cdot\|_\infty$ to be called a norm. However, there is a subtle difference between statement (b) in the next exercise and the uniqueness property of a norm given in Definition 0.1. We will discuss this difference in more detail after the exercise. To avoid technical complications, we will assume that our measure space is complete and scalars are complex in this exercise.

Exercise 3.55. Let (X, Σ, μ) be a measure space, and let $\mathcal{E}_b(X)$ be the set of all measurable, essentially bounded, complex-valued functions on X, i.e.,

$$\mathcal{E}_b(X) \;=\; \bigl\{f \colon X \to \mathbb{C} \,:\, f \text{ is measurable and essentially bounded}\bigr\}.$$

It follows from Exercise 3.45 that $\mathcal{E}_b(X)$ is a vector space over the complex field. Show that uniform norm satisfies the following properties on $\mathcal{E}_b(X)$:

(a) $0 \le \|f\|_\infty \le \infty$ for all $f \in \mathcal{E}_b(X)$,

(b) $\|f\|_\infty = 0$ if and only if $f = 0$ a.e.,

(c) $\|cf\|_\infty = |c|\,\|f\|_\infty$ for all $f \in \mathcal{E}_b(X)$ and scalars $c \in \mathbb{C}$,

(d) $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ for all $f,\, g \in \mathcal{E}_b(X)$. ♢

Unfortunately, statement (b) in Exercise 3.55 is not quite what we need in order to say that $\|\cdot\|_\infty$ is a norm on $\mathcal{E}_b(X)$. According to Definition 0.1, to be a norm it must be the case that $\|f\|_\infty = 0$ if and only if f is the zero vector in $\mathcal{E}_b(X)$. The zero vector in $\mathcal{E}_b(X)$ is the zero function, the function that takes the value 0 for every $x \in X$. However, it is not usually true that $\|f\|_\infty = 0$ if and only if f is the zero function. Instead we only have the weaker statement that

$$\|f\|_\infty = 0 \qquad \Longleftrightarrow \qquad f = 0 \text{ a.e.} \tag{3.9}$$

The technical terminology for this is that $\|\cdot\|_\infty$ is a *seminorm* on $\mathcal{E}_b(X)$ rather than a *norm*. Since our philosophy is that sets of measure zero usually "don't matter," this is not really a problem—we simply accept the fact that the right-hand side of equation (3.9) says that f is zero almost everywhere, instead of saying that f is zero everywhere. Problem 3.21 shows one way to modify the space on which $\|\cdot\|_\infty$ is defined so that it becomes a true norm on that space. This approach will be useful to us when we define the L^p spaces in Chapter 5, but right now it is not important.

Example 3.56. Consider $X = \mathbb{R}$ and $f = \chi_C$, the characteristic function of the Cantor set C. Since the Cantor zero has measure zero, its characteristic function satisfies $\chi_C = 0$ a.e. Therefore $\|\chi_C\|_\infty = 0$, even though χ_C is not the zero function. ♢

Whenever we have a norm (or a seminorm), it is very important to know whether Cauchy sequences must converge with respect to that norm. Following the usual terminology for norms (Definition 0.2), we say that a sequence $\{f_k\}_{k\in\infty}$ in $\mathcal{E}_b(X)$ is *Cauchy in L^∞ norm* or *Cauchy with respect to* $\|\cdot\|_\infty$ if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, j, k \ge N, \quad \|f_j - f_k\|_\infty < \varepsilon.$$

We will show in the next theorem that every Cauchy sequence in $\mathcal{E}_b(X)$ must converge in $\mathcal{E}_b(X)$, at least if our measure μ is a complete measure on X. If $\|\cdot\|_\infty$ was a true norm on $\mathcal{E}_b(X)$ then we would describe this by saying that

$\mathcal{E}_b(X)$ is a *complete normed space* (see Definition 0.3). However, since $\|\cdot\|_\infty$ is not a true norm, we will not use that terminology in connection with $\mathcal{E}_b(X)$.

Theorem 3.57. *Let (X, Σ, μ) be a complete measure space. If $\{f_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{E}_b(X)$ that is Cauchy with respect to $\|\cdot\|_\infty$, then there exists a function $f \in \mathcal{E}_b(X)$ such that $\|f - f_k\|_\infty \to 0$ as $k \to \infty$.*

Proof. Let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{E}_b(X)$. This means that

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, j, k \geq N, \quad \|f_j - f_k\|_\infty < \varepsilon.$$

Let $M_{jk} = \|f_j - f_k\|_\infty$. Then $|f_j(x) - f_k(x)|$ can exceed M_{jk} only for a set of x's that lie in a set of measure zero. That is, if we define

$$Z_{jk} \;=\; \bigl\{|f_j - f_k| > M_{jk}\bigr\},$$

then $\mu(Z_{jk}) = 0$. Since a countable union of zero measure sets has measure zero, it follows that

$$Z \;=\; \bigcup_{j,k\in\mathbb{N}} Z_{jk}$$

has measure zero. For each $k \in \mathbb{N}$, create a function g_k that equals f_k almost everywhere but is zero on the set Z:

$$g_k(x) \;=\; \begin{cases} f_k(x), & x \notin Z, \\ 0, & x \in Z. \end{cases}$$

Since μ is a complete measure, Lemma 3.27 implies that g_k is a measurable function. Exercise: Show that the uniform norm of $g_j - g_k$ coincides with its L^∞ norm, which also coincides with the L^∞ norm of $f_j - g_k$, i.e.,

$$\|g_j - g_k\|_{\mathrm{u}} \;=\; \|g_j - g_k\|_\infty \;=\; \|f_j - f_k\|_\infty.$$

Consequently, $\{g_k\}_{k\in\mathbb{N}}$ is a uniformly Cauchy in $\mathcal{F}_b(X)$, the space of bounded, complex-valued functions on X. Theorem 0.12 therefore implies that there is a bounded function g such that $g_k \to g$ uniformly. This function g is measurable since it is the pointwise limit of measurable functions. Hence g belongs to $\mathcal{E}_b(X)$. Finally, since $g - g_k$ differs from $g - f_k$ only on a set of measure zero, we have

$$\|g - f_k\|_\infty \;=\; \|g - g_k\|_\infty \;\leq\; \|g - g_k\|_{\mathrm{u}} \;\to\; 0 \quad \text{as } k \to \infty. \qquad \diamondsuit$$

Example 3.58. Let $Z_1 \subseteq Z_2 \subseteq \cdots$ be a nested increasing sequence of subsets of $\mathbb{R}$ that each have measure zero. Define

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Note the two distinct uses of the word "complete" in this paragraph! A *complete measure* is a measure such that every subset of a null set is measurable, while a *complete normed space* is a space such that every Cauchy sequence converges.

$$f_k(x) \;=\; \begin{cases} \frac{1}{k}, & x \notin Z_k, \\ k, & x \in Z_k. \end{cases}$$

The functions f_k do not converge pointwise to zero since there points x where $f_k(x) \to \infty$ as k increases. On the other hand, f_k converges to the zero function with respect to $\|\cdot\|_\infty$, because

$$\|0 - f_k\|_\infty \;=\; \|f_k\|_\infty \;=\; \frac{1}{k} \quad \text{as } k \to \infty. \qquad \diamondsuit$$

If we like, we could make analogous versions of Exercise 3.55 and Theorem 3.57 for extended real-valued functions that are finite almost everywhere.

3.8.3 Convergence in Measure

Pointwise convergence, pointwise almost everywhere convergence, uniform convergence, and L^∞ norm convergence are only some of the myriad ways in which we might say that functions f_k "converge" to a given function f. For pointwise convergence we require that $f_k(x)$ becomes close to $f(x)$ for each individual x, while for uniform convergence we require that $f_k(x)$ and $f(x)$ become simultaneously close over all x. Pointwise almost everywhere and L^∞ norm convergence are variations where we ignore sets of measure zero. Now we describe another type of convergence criterion that we will call convergence in measure.

Suppose that we have functions f_k and f on a measure space (X, Σ, μ). The idea of convergence in measure is that we require $f_k(x)$ and $f(x)$ to be close except on a set that has smaller and smaller measure. More precisely, given any fixed $\varepsilon > 0$, the set where $f_k(x)$ and $f(x)$ differ by more than ε should become smaller and smaller in measure as k increases. Here is the explicit definition for complex-valued functions.

Definition 3.59 (Convergence in Measure). Let (X, Σ, μ) be a measure space, and let f_k, f be complex-valued measurable functions on E. Then we say that f_k *converges in measure to* f, and write $f_k \xrightarrow{\text{m}} f$, if

$$\forall\, \varepsilon > 0, \quad \lim_{k\to\infty} \mu\bigl\{|f - f_k| > \varepsilon\bigr\} \;=\; 0. \qquad \diamondsuit \tag{3.10}$$

It is useful to write out equation (3.10) in complete detail. With all the quantifiers, the requirement for convergence in measure given in (3.10) is that

$$\forall\, \varepsilon, \eta > 0, \quad \exists\, N > 0 \;\text{ such that }\; k > N \;\implies\; \mu\{|f - f_k| \ge \varepsilon\} \;<\; \eta. \tag{3.11}$$

Problem 3.19 gives an equivalent formulation that requires only the use of a single variable ε instead of two variables ε, η.

We can make a similar definition for convergence in measure of extended real-valued functions, but there is a technical complication because $f - f_k$ might not be measurable. However, if our measure space is complete and the functions f_k and f are finite a.e. then this is not a problem, so we extend the definition to this case as follows.

Definition 3.60 (Convergence in Measure). Let (X, Σ, μ) be a complete measure space, and let f_k, f be extended real-valued measurable functions on E that are finite a.e. Then we say that f_k *converges in measure to* f, and write $f_k \xrightarrow{\mathrm{m}} f$, if

$$\forall\, \varepsilon > 0, \quad \lim_{k \to \infty} \mu\bigl\{|f - f_k| > \varepsilon\bigr\} \;=\; 0. \qquad \diamondsuit$$

For the remainder of this section we will state results for the case of complex-valued functions, and assign the reader the task of formulating analogous results for extended real-valued functions.

Since $\{|f - f_k| > \varepsilon\}$ is the set where f and f_k differ by more than ε, these definitions are saying that the measure of this set must decrease to zero as k increases. For example, if $f_k = f$ except on a set E_k of measure $1/k$, then $f_k \xrightarrow{\mathrm{m}} f$. This example suggests that convergence in measure might imply pointwise or pointwise almost everywhere convergence, but the following exercises and examples show that the relationships among these types of convergence are not quite as straightforward as we might hope.

Exercise 3.61. Show that L^∞ norm convergence implies both pointwise a.e. convergence and convergence in measure. That is, if (X, Σ, μ) is a measure space and $\|f - f_k\|_\infty \to 0$, then $f_k \to f$ pointwise a.e. and also $f_k \xrightarrow{\mathrm{m}} f$. $\diamondsuit$

Example 3.62 (Shrinking Boxes). Let $f_k = \chi_{[0,\frac{1}{k}]}$, the characteristic function of the closed interval $[0, \frac{1}{k}]$. As we saw in Example 3.51, these functions converge pointwise a.e. to the zero function. However, since $\|f_k\|_\infty = 1$ for every k, the functions f_k do not converge in L^∞ norm to the zero function. Hence pointwise a.e. convergence does not imply L^∞ norm convergence.

However, this sequence does converge in measure to the zero function. To see this, fix any $0 < \varepsilon < 1$. The set where f_k differs from the zero function by more than ε is precisely the interval $[0, \frac{1}{k}]$. The Lebesgue measure of this set converges to zero as k increases, so $f_k \xrightarrow{\mathrm{m}} 0$. $\diamondsuit$

A variation on the preceding example is to take $g_k = k \cdot \chi_{[0,\frac{1}{k}]}$. Even though the heights of these functions grow without bound as k increases, we still have that $g_k \to 0$ a.e. and $g_k \xrightarrow{\mathrm{m}} 0$.

Example 3.63 (Boxes Marching to Infinity). This example will show that pointwise a.e. convergence does not imply convergence in measure. Let $f_k = \chi_{[k,k+1]}$, the characteristic function of the closed interval $[k, k+1]$. Then we have that $f_k \to 0$ pointwise, although we do not have L^∞ convergence to the zero function because $\|f_k\|_\infty = 1$ for every k.

To see that we do not have convergence in measure, fix any $0 < \varepsilon < 1$. The set where f_k differs from 0 by more than ε is the interval $[k, k+1]$, which has Lebesgue measure 1 for every k. Since this quantity does not shrink to zero, f_k does not converge in measure to the zero function. $\diamondsuit$

Example 3.64 (Boxes Marching in Circles). This example will show that convergence in measure does not imply pointwise a.e. convergence. Define

$$\begin{aligned} &f_1 = \chi_{[0,1]}, \\ &f_2 = \chi_{[0,\frac{1}{2}]}, \quad f_3 = \chi_{[\frac{1}{2},1]}, \\ &f_4 = \chi_{[0,\frac{1}{3}]}, \quad f_5 = \chi_{[\frac{1}{3},\frac{2}{3}]}, \quad f_6 = \chi_{[\frac{2}{3},1]}, \\ &f_7 = \chi_{[0,\frac{1}{4}]}, \quad f_8 = \chi_{[\frac{1}{4},\frac{1}{2}]}, \quad f_9 = \chi_{[\frac{1}{2},\frac{3}{4}]}, \quad f_{10} = \chi_{[\frac{3}{4},1]}, \end{aligned}$$

and so forth. Picturing the graphs of these functions as boxes, the boxes march from left to right across the interval $[0,1]$, then shrink in size and march across the interval again, and do this over and over.

Fix any $0 < \varepsilon < 1$. For the indices $k = 1, \dots, 10$, the Lebesgue measure of $\{|f_k| > \varepsilon\}$ is

$$1, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{3}, \ \frac{1}{3}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4}.$$

The Lebesgue measure of $\{|f_k| > \varepsilon\}$ converges to zero as k increases, so $f_k \xrightarrow{\mathrm{m}} 0$.

We do not have pointwise a.e. convergence in this example because no matter what point $x \in [0,1]$ that we choose, there are infinitely many different values of k such that $f_k(x) = 0$ and infinitely many k such that $f_k(x) = 1$. While $f_k(x)$ may be zero for many particular k, there are always larger k such that $f_k(x) = 1$. Hence $f_k(x)$ does not converge to 0 for any $x \in [0,1]$. This sequence of functions *does not* converge pointwise (or even pointwise a.e.) to the zero function, or to any other function. ♢

In summary, convergence in L^∞ norm implies pointwise a.e. convergence and convergence in measure, but pointwise a.e. convergence does not imply convergence in measure, and convergence in measure does not imply pointwise a.e. convergence. Just to complicate matters, after we prove Egoroff's Theorem (Theorem 3.68) we will be able to show that, *in a finite measure space*, pointwise a.e. convergence implies convergence in measure (see Problem 3.23).

In the converse direction, Example 3.64 shows us that convergence in measure does not imply pointwise a.e. convergence, even in a finite measure space. However, there is a weaker but still very important relation between convergence in measure and pointwise a.e. convergence. Specifically, the next theorem will show that if f_k converges in measure to f, then there is a *subsequence* of the f_k that converges pointwise almost everywhere to f. For example, $\{\chi_{[0,\frac{1}{k}]}\}_{k\in\mathbb{N}}$ is a subsequence of the Marching Boxes from Example 3.64, and $\chi_{[0,\frac{1}{k}]}$ converges pointwise a.e. to the zero function.

Theorem 3.65. *Let (X, Σ, μ) be a measure space. If $f_k,\, f \colon X \to \mathbb{C}$ are measurable functions such that $f_k \xrightarrow{\mathrm{m}} f$, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ pointwise a.e.*

Proof. Assume $f_k \xrightarrow{\text{m}} f$. Exercise: Show that there exist indices $n_1 < n_2 < \cdots$ such that

$$\mu\big\{|f - f_{n_k}| > \tfrac{1}{k}\big\} \;\le\; 2^{-k}, \qquad k \in \mathbb{N}.$$

Define

$$E_k \;=\; \big\{|f - f_{n_k}| > \tfrac{1}{k}\big\} \qquad \text{and} \qquad H_m \;=\; \bigcup_{k=m}^{\infty} E_k.$$

These sets are measurable since f_k and f are measurable functions. By construction we have $\mu(E_k) \le 2^{-k}$, so by subadditivity,

$$\mu(H_m) \;\le\; \sum_{k=m}^{\infty} \mu(E_k) \;\le\; \sum_{k=m}^{\infty} \frac{1}{2^k} \;=\; 2^{-m+1}.$$

Set

$$Z \;=\; \bigcap_{m=1}^{\infty} H_m.$$

For each $m \in \mathbb{N}$ we have $\mu(Z) \le \mu(H_m) \le 2^{-m+1}$, so Z has measure zero (note that $Z = \limsup E_k$ in the terminology of Problem 2.18).

If $x \notin Z$, then $x \notin H_m$ for some m. Hence $x \notin E_k$ for all $k \ge m$, which implies that

$$|f(x) - f_{n_k}(x)| \;\le\; \frac{1}{k} \quad \text{for all } k \ge m.$$

Thus $f_{n_k}(x) \to f(x)$ when $x \notin Z$. Since Z has measure zero, this says that the functions f_{n_k} converge pointwise almost everywhere to f. □

Unlike L^∞ convergence, it is not possible to characterize convergence in measure in terms of a norm. Problem 3.22 shows that it is possible to create a *metric* that corresponds to convergence in measure, but in practice it is usually more convenient work directly with the definition of convergence in measure instead of trying to use that metric. On the other hand, it is important to know whether there is a notion of sequences that are Cauchy with respect to convergence in measure, and whether such Cauchy sequences must converge in measure. The next definition introduces the Cauchy criterion.

Definition 3.66 (Cauchy in Measure). Let (X, Σ, μ) be a measure space. Given measurable functions $f_k,\, f\colon X \to \mathbb{C}$, we say that $\{f_k\}_{k\in\mathbb{N}}$ is *Cauchy in measure* if

$$\forall\, \varepsilon > 0, \qquad \lim_{j,k\to\infty} \mu\{|f_j - f_k| > \varepsilon\} \;=\; 0. \tag{3.12}$$

Precisely, equation (3.12) means that

$$\forall\, \varepsilon, \eta > 0, \quad \exists\, N > 0 \;\text{ such that }\; j, k > N \;\implies\; \mu\{|f_j - f_k| > \varepsilon\} \;<\; \eta. \quad \diamondsuit$$

Now we prove that if a sequence is Cauchy in measure then it must converge in measure. The usefulness of the Cauchy criterion is that we can test for it without knowing what the limit function is, or even whether it exists.

Theorem 3.67 (Cauchy Criterion for Convergence in Measure). *Let (X, Σ, μ) be a measure space. If $f_k\colon X \to \mathbb{C}$ are measurable functions on X, then the following statements are equivalent.*

(a) *There exists a measurable function f such that $f_k \xrightarrow{\text{m}} f$.*

(b) *$\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in measure.*

Proof. (a) $\Rightarrow$ (b). Assume that $f_k \xrightarrow{\text{m}} f$, and fix $\varepsilon, \eta > 0$. Then by equation 3.11, there exists an $N > 0$ such that

$$\mu\{|f - f_k| > \varepsilon\} \;<\; \eta \qquad \text{for all } k > N.$$

By the Triangle Inequality,

$$|f_j(x) - f_k(x)| \;\le\; |f_j(x) - f(x)| + |f(x) - f_k(x)|.$$

It follows from this that

$$\{|f_j - f_k| > 2\varepsilon\} \;\subseteq\; \{|f_j - f| > \varepsilon\} \,\cup\, \{|f - f_k| > \varepsilon\}.$$

Consequently, if $k > N$ then

$$\mu\{|f_j - f_k| > 2\varepsilon\} \;\le\; \mu\{|f_j - f| > \varepsilon\} \;+\; \mu\{|f - f_k| > \varepsilon\} \;<\; \eta + \eta \;=\; 2\eta.$$

Therefore $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in measure.

(b) $\Rightarrow$ (a). The first part of this proof proceeds much like the proof of Theorem 3.65. Suppose that $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in measure. Then there exist indices $n_1 < n_2 < \cdots$ such that

$$\mu\bigl\{|f_{n_{k+1}} - f_{n_k}| > 2^{-k}\bigr\} \;\le\; 2^{-k}, \qquad k \in \mathbb{N}.$$

For simplicity of notation, let $g_k = f_{n_k}$. Define

$$E_k \;=\; \bigl\{|g_{k+1} - g_k| > 2^{-k}\bigr\} \qquad \text{and} \qquad H_m \;=\; \bigcup_{k=m}^{\infty} E_k.$$

Then $\mu(E_k) \le 2^{-k}$, so

$$\mu(H_m) \;\le\; \sum_{k=m}^{\infty} \mu(E_k) \;\le\; \sum_{k=m}^{\infty} 2^{-k} \;=\; 2^{-m+1}.$$

Therefore, if we set

$$Z \;=\; \bigcap_{m=1}^{\infty} H_m \;=\; \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k \;=\; \limsup E_k,$$

then Z is measurable and $\mu(Z) = 0$.

If $x \notin Z$ then $x \notin H_m$ for some m, and therefore $x \notin E_k$ for all $k \geq m$. That is, $|g_{k+1}(x) - g_k(x)| \leq 2^{-k}$ for all $k \geq m$. This implies that $\{g_k(x)\}_{k\in\mathbb{N}}$ is a Cauchy sequence of complex scalars (compare Problem 0.3), and therefore must converge. The function

$$f(x) \;=\; \begin{cases} \lim\limits_{k\to\infty} g_k(x), & \text{if the limit exists,} \\ 0, & \text{otherwise,} \end{cases}$$

is measurable by Corollary 3.48, and we have by construction that $g_k \to f$ pointwise a.e.

Now will show that g_k converges in measure to f. Fix $\varepsilon > 0$, and choose m large enough that $2^{-m} \leq \varepsilon$. Fix $m \in \mathbb{N}$, and suppose that $x \notin H_m$. If $n > k > m$ then we have

$$|g_n(x) - g_k(x)| \;\leq\; \sum_{i=k}^{n-1} |g_{i+1}(x) - g_i(x)| \;\leq\; \sum_{i=k}^{n-1} 2^{-i} \;\leq\; 2^{-k+1} \;\leq\; 2^{-m} \;\leq\; \varepsilon.$$

Taking the limit as $n \to \infty$, this implies that $|f(x) - g_k(x)| \leq \varepsilon$ for all $x \notin H_m$ and $k > m$. Hence

$$\bigl\{|f - g_k| > \varepsilon\bigr\} \;\subseteq\; H_m, \qquad k > m,$$

so

$$\limsup_{k\to\infty} \mu\bigl\{|f - g_k| > \varepsilon\bigr\} \;\leq\; \mu(H_m) \;\leq\; 2^{-m+1}.$$

This is true for every m, so we conclude that $\lim_{k\to\infty} \mu\{|f - g_k| > \varepsilon\} = 0$. Therefore $g_k \xrightarrow{\text{m}} f$. This is not quite enough to complete the proof, becuase $\{g_k\}_{k\in\mathbb{N}}$ is only a subsequence of $\{f_k\}_{k\in\mathbb{N}}$. However, by Problem 3.18, the fact that $\{f_k\}_{k\in N}$ is Cauchy in measure and has a subsequence that converges in measure to f implies that $f_k \xrightarrow{\text{m}} f$. $\square$

Additional Problems

3.18. Let (X, Σ, μ) be a measure space, and let $f_k,\, f\colon X \to \mathbb{C}$ be measurable functions on X. Show that if $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in measure and there exists a subsequence such that $f_{n_k} \xrightarrow{\text{m}} f$, then $f_k \xrightarrow{\text{m}} f$.

3.19. Let (X, Σ, μ) be a measure space, and let $f_k,\, f\colon X \to \mathbb{C}$ be measurable functions on X. Prove that $f_k \xrightarrow{\text{m}} f$ if and only if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0 \;\text{ such that }\; k > N \;\implies\; \mu\{|f - f_k| \geq \varepsilon\} \;<\; \varepsilon.$$

Formulate and prove an analogous Cauchy criterion.

3.20. Let (X, Σ, μ) be a measure space. Let $f_k, f, g_k, g\colon X \to \mathbb{C}$ be measurable functions on X, and prove the following statements.

(a) If $f_k \overset{\mathrm{m}}{\to} f$ and $f_k \overset{\mathrm{m}}{\to} g$, then $f = g$ a.e.

(b) If $f_k \overset{\mathrm{m}}{\to} f$ and $g_k \overset{\mathrm{m}}{\to} g$, then $f_k + g_k \overset{\mathrm{m}}{\to} f + g$.

(c) If μ is a bounded measure, $f_k \overset{\mathrm{m}}{\to} f$, and $g_k \overset{\mathrm{m}}{\to} g$, then $f_k g_k \overset{\mathrm{m}}{\to} fg$.

(d) The conclusion of part (c) can fail if μ is not bounded.

3.21. Let (X, Σ, μ) be a measure space.

(a) Show that $f \sim g$ if $f = g$ a.e. is an equivalence relation on $\mathcal{E}_b(X)$.

(b) Let $\widetilde{f}$ denote the equivalence class of f under this relation, i.e.,

$$\widetilde{f} \;=\; \bigl\{g \in \mathcal{E}_b(X) \,:\, g = f \text{ a.e.}\bigr\}.$$

Define $\|\!|\widetilde{f}\|\!|_\infty = \|f\|_\infty$, and show that this quantity is independent of the choice of representative f of $\widetilde{f}$.

(c) Let the quotient space $L^\infty(X)$ consist of all the distinct equivalence classes of $f \in \mathcal{E}_b(X)$, i.e.,

$$L^\infty(X) \;=\; \bigl\{\widetilde{f} : f \in \mathcal{E}_b(X)\bigr\}.$$

Define operations of addition and scalar multiplication that make $L^\infty(X)$ into a vector space over the complex field. What is the zero element of $L^\infty(X)$?

(d) Show that $\|\!|\cdot\|\!|_\infty$ is a norm on $L^\infty(X)$.

3.22. Let (X, Σ, μ) be a measure space, and let $\mathcal{F}_{\mathrm{m}}(X)$ be the vector space of all complex-valued measurable functions on X. For each $n \in \mathbb{N}$, define

$$\rho_n(f) \;=\; \mu\bigl\{|f| > \tfrac{1}{n}\bigr\}$$

and for $f, g \in \mathcal{F}_{\mathrm{m}}(X)$ define

$$\mathrm{d}(f,g) \;=\; \sum_{n=1}^{\infty} 2^{-n}\, \frac{\rho_n(f-g)}{1+\rho_n(f-g)}.$$

Prove that $\rho_n(f+g) \le \rho_{2n}(f) + \rho_{2n}(g)$, and use this to prove the following statements.

(a) $\mathrm{d}(f,g) \ge 0$,

(b) $\mathrm{d}(f,g) = 0$ if and only if $f = g$,

(c) $\mathrm{d}(f,g) = \mathrm{d}(g,f)$, and

(d) $\mathrm{d}(f,h) \le \mathrm{d}(f,g) + \mathrm{d}(g,h)$.

Consequently, if we identify functions that are equal almost everywhere then d is a metric on $\mathcal{F}_{\mathrm{m}}(x)$. Prove that convergence with respect to this metric coincides with convergence in measure, i.e.,

$$f_k \overset{\mathrm{m}}{\to} f \quad\Longleftrightarrow\quad \lim_{k\to\infty} \mathrm{d}(f, f_k) \;=\; 0.$$