2.5 Outer Measures

In Chapter 1, our goal was to extend the notion of measure from volumes of boxes to arbitrary subsets of $\mathbb{R}^d$. We could not do this in a "good" way for all sets, so we constructed exterior Lebesgue measure on the measure space $(\mathbb{R}^d, \mathcal{P}(\mathbb{R}^d))$, and then restricted to the class of Lebesgue measurable subsets in order to obtain Lebesgue measure. In contrast, this chapter deals with abstract measures which, by definition, satisfy countable additivity on the σ-algebra with the measure. But this leaves us with an important question: How can we construct an abstract measure? The goal of this section and the next is generalize the idea of deriving Lebesgue measure from exterior Lebesgue measure to an abstract setting. Specifically, given a class of subsets of X that we know how we want to measure, we will construct an exterior or outer measure μ^* that is defined on all subsets of X but is not a true measure, and then construct a actual measure μ by restricting μ^* to an appropriate σ-algebra Σ of "measurable sets."

It will be easier if we break this process into two parts:

(i) the construction of an outer measure μ^*, and

(ii) the construction of a measure μ from the outer measure μ^*.

It will also be conceptually more clear if we tackle the second item first. So, in this section we show how to obtain a measure μ from an outer measure μ^*, while in the next section we will consider the issue of constructing an outer measure μ^* from scratch.

Our first task is to precisely define outer measures. Considering the example of exterior Lebesgue measure, it seems that the most important requirements are that an outer measure μ^* should be defined on all subsets of X and that it should satisfy countable subadditivity. However, there is another important but hidden property: Monotonicity. Although *countable additivity* implies monotonicity, *countable subadditivity* does not. Therefore we need to include monotonicity as part of the definition of an outer measure.

Definition 2.28 (Outer Measure). Let X be a nonempty set. An *outer measure* or *exterior measure* on X is a function $\mu^* \colon \mathcal{P}(X) \to [0,\infty]$ that satisfies the following conditions.

(a) $\mu^*(\emptyset) = 0$.

(b) *Monotonicity*: If $A \subseteq B$, then $\mu^*(A) \le \mu^*(B)$.

(c) *Countable subadditivity*: If $E_1, E_2, \ldots \subseteq X$, then

$$\mu^*\Big(\bigcup_k E_k\Big) \;\le\; \sum_k \mu^*(E_k). \qquad \diamondsuit$$

Given an outer measure μ^*, our goal is to create a σ-algebra Σ on X such that μ^* *restricted to* Σ will be countably additive. The elements of Σ will be our "good sets," the sets that are measurable with respect to μ^*. But how do we define measurability for an arbitrary outer measure? We might not be given a topology on X, so we cannot define measurability in terms of surrounding open sets, as we did for Lebesgue measure (Definition 1.19). However, the formulation of Lebesgue measurability given by Carathéodory's Criterion (Theorem 1.45) does not involve topology, and as such it is the appropriate motivation for the following definition.

Definition 2.29 (Measurable Set). Let μ^* be an outer measure on a set X. Then a set $E \subseteq X$ is *μ^*-measurable*, or simply *measurable* for short, if

$$\forall A \subseteq X, \quad \mu^*(A) \;=\; \mu^*(A \cap E) + \mu^*(A \backslash E). \qquad \diamondsuit$$

Remark 2.30. (a) It is often helpful to recall that $\mu^*(A\backslash E) = \mu^*(A \cap E^{\mathrm{C}})$, so we can equivalently write the condition for measurability as

$$\forall A \subseteq X, \quad \mu^*(A) \;=\; \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{C}}).$$

(b) The empty set is μ^*-measurable by virtue of the fact that $\mu^*(\emptyset) = 0$.

(c) By subadditivity, we always have the inequality

$$\mu^*(A) \;\le\; \mu^*(A \cap E) + \mu^*(A \backslash E).$$

Hence, to establish that a set E is measurable, we just have to prove that the opposite inequality holds for every set $A \subseteq X$. $\diamondsuit$

The next exercise, which follows by combining subadditivity with monotonicity, asks for a proof that every subset of X that has outer measure zero is measurable.

Exercise 2.31. Let μ^* be an outer measure on X. Show that if $E \subseteq X$ and $\mu^*(E) = 0$ then E is μ^*-measurable. $\diamondsuit$

Now we will prove that the collection of μ-measurable sets forms a σ-algebra on X, and μ^* restricted to this σ-algebra is a complete measure on X.

Theorem 2.32 (Carathéodory's Theorem). *If μ^* is an outer measure on a set X, then the following statements hold.*

(a) *The family*

$$\Sigma \;=\; \bigl\{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\bigr\}$$

is a σ-algebra on X.

(b) $\mu = \mu^*|_\Sigma$ *is a measure on* (X, Σ).

(c) μ *is a complete measure. In fact,* every *set $E \subseteq X$ that satisfies $\mu^*(E) = 0$ is μ^*-measurable.*

Proof. (a) Σ is not empty since the empty set is μ^*-measurable.

To show that Σ is closed under complements, fix any set $E \in \Sigma$ and let A be an arbitrary subset of X. Using the fact that E is μ^*-measurable, we compute that

$$\begin{aligned}
\mu^*(A) &= \mu^*(A \cap E) + \mu^*(A \backslash E) \\
&= \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{C}}) \\
&= \mu^*(A \cap E^{\mathrm{C}}) + \mu^*\big(A \cap (E^{\mathrm{C}})^{\mathrm{C}}\big) \\
&= \mu^*(A \cap E^{\mathrm{C}}) + \mu^*(A \backslash E^{\mathrm{C}}).
\end{aligned}$$

Hence E^{C} is μ^*-measurable, so $E^{\mathrm{C}} \in \Sigma$.

Ultimately we want to show that Σ is closed under countable unions, but to begin we will show that it is closed under finite unions. By induction, to prove this it suffices to show that if $E,\ F \in \Sigma$ then $E \cup F \in \Sigma$. Choose any set $A \subseteq X$. By subadditivity,

$$\mu^*(A) \;\le\; \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cap F)^{\mathrm{C}}).$$

To prove the opposite inequality, note that since F is μ^*-measurable and $A \cap E$ is a subset of X, we have

$$\mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^{\mathrm{C}}) \;=\; \mu^*(A \cap E).$$

Similarly,

$$\mu^*(A \cap E^{\mathrm{C}} \cap F) + \mu^*(A \cap E^{\mathrm{C}} \cap F^{\mathrm{C}}) \;=\; \mu^*(A \cap E^{\mathrm{C}}).$$

Applying these equalities and using the fact that μ^* is finitely subadditive, we see that

$$\begin{aligned}
&\mu^*(A \cap (E \cup F)) \;+\; \mu^*(A \cap (E \cup F)^{\mathrm{C}}). \\
&\quad = \mu^*\big((A \cap E \cap F) \cup (A \cap E \cap F^{\mathrm{C}}) \cup (A \cap E^{\mathrm{C}} \cap F)\big) \;+\; \mu^*(A \cap (E \cup F)). \\
&\quad \le \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^{\mathrm{C}}) + \mu^*(A \cap E^{\mathrm{C}} \cap F) + \mu^*(A \cap E^{\mathrm{C}} \cap F^{\mathrm{C}}) \\
&\quad = \mu^*(A \cap E) \;+\; \mu^*(A \cap E^{\mathrm{C}}) \\
&\quad = \mu^*(A),
\end{aligned}$$

where at the final equality we have used the fact that E is μ^*-measurable. Therefore $E \cup F \in \Sigma$.

Before proceeding, we make an observation. Suppose that $E,\ F \in \Sigma$ are *disjoint*. Since E is measurable, we have

$$\mu^*(E \cup F) \;=\; \mu^*\big((E \cup F) \cap E\big) + \mu^*\big((E \cup F) \cap E^{\mathrm{C}}\big) \;=\; \mu^*(E) + \mu^*(F).$$

Thus, μ^* is *finitely additive on* Σ. However, μ^* need not be finitely additive on arbitrary subsets of X!

Now we will show that Σ is closed under countable unions. By Problem 2.4, it suffices to show that Σ is closed under countable *disjoint* unions. So, assume that $E_1, E_2, \ldots \in \Sigma$ are disjoint sets and define

$$F \;=\; \bigcup_{k=1}^{\infty} E_k \qquad \text{and} \qquad F_n \;=\; \bigcup_{k=1}^{n} E_k, \quad n \in \mathbb{N}.$$

Note that $F_n \in \Sigma$ since Σ is closed under finite unions.

Choose any set $A \subseteq X$. We claim that

$$\mu^*(A \cap F_n) \;=\; \sum_{k=1}^{n} \mu^*(A \cap E_k). \tag{2.4}$$

Now, if we knew that μ^* was finitely additive on all subsets of X then equation (2.4) would be immediate. However, we only know that μ^* is finitely additive on the μ^*-measurable sets. Since A is an arbitrary set, this does not help us. Instead, we prove equation (2.4) by induction.

Since $F_1 = E_1$, equation (2.4) is trivial when $n = 1$. Suppose that equation (2.4) holds for some integer $n \geq 1$. Then, since E_{n+1} is μ^*-measurable,

$$\begin{aligned}
\mu^*(A \cap F_{n+1}) \;&=\; \mu^*\Big(A \,\cap\, \bigcup_{k=1}^{n+1} E_k\Big) \\
&=\; \mu^*\Big(A \,\cap\, \bigcup_{k=1}^{n+1} E_k \,\cap\, E_{n+1}\Big) + \mu^*\Big(A \,\cap\, \bigcup_{k=1}^{n+1} E_k \,\cap\, E_{n+1}^{\mathrm{C}}\Big) \\
&=\; \mu^*(A \cap E_{n+1}) + \mu^*(A \,\cap\, \bigcup_{k=1}^{n} E_n) \qquad \text{(by disjointness)} \\
&=\; \mu^*(A \cap E_{n+1}) + \sum_{k=1}^{n} \mu^*(A \cap E_k).
\end{aligned}$$

It follows that equation (2.4) holds for all n by induction.

Next we compute that

$$\begin{aligned}
&\sum_{k=1}^{n} \mu^*(A \cap E_k) \;+\; \mu^*(A \cap F^{\mathrm{C}}) \\
&\quad\leq\; \sum_{k=1}^{n} \mu^*(A \cap E_k) \;+\; \mu^*(A \cap F_n^{\mathrm{C}}) \qquad \text{(since } F^{\mathrm{C}} \subseteq F_n^{\mathrm{C}}) \\
&\quad=\; \mu^*(A \cap F_n) + \mu^*(A \cap F_n^{\mathrm{C}}) \qquad \text{(by equation (2.4))} \\
&\quad=\; \mu^*(A) \qquad \text{(since } F_n \in \Sigma).
\end{aligned} \tag{2.5}$$

Applying subadditivity and taking the limit as $n \to \infty$,

$$\begin{aligned}
\mu^*(A) &\le \mu^*(A \cap F) + \mu^*(A \cap F^{\mathrm{C}}) && \text{(subadditivity)} \\
&\le \sum_{k=1}^{\infty} \mu^*(A \cap E_k) \;+\; \mu^*(A \cap F^{\mathrm{C}}) && \text{(subadditivity)} \\
&\le \mu^*(A) && \text{(by equation (2.5)).}
\end{aligned}$$

Therefore equality holds in the preceding lines:

$$\begin{aligned}
\mu^*(A) &= \sum_{k=1}^{\infty} \mu^*(A \cap E_k) \;+\; \mu^*(A \cap F^{\mathrm{C}}) \qquad (2.6) \\
&= \mu^*(A \cap F) + \mu^*(A \cap F^{\mathrm{C}}),
\end{aligned}$$

so $F \in \Sigma$.

(b) To show that μ^* *restricted to* Σ is countably additive, let $E_1, E_2, \dots$ be disjoint sets in Σ, and set $F = \cup E_k$. Then since F is μ^*-measurable, by applying equation (2.6) with $A = F$ we see that

$$\mu^*(F) \;=\; \sum_{k=1}^{\infty} \mu^*(F \cap E_k) \;+\; \mu^*(F \cap F^{\mathrm{C}}) \;=\; \sum_{k=1}^{\infty} \mu^*(E_k).$$

Hence μ^* is countably additive on Σ, and therefore $\mu = \mu^*|_{\Sigma}$ is a measure.

(c) This follows from Exercise 2.31. $\square$

Additional Problems

2.21. Let μ^* be an outer measure on X, and let A, $B \subseteq X$ be μ^*-measurable. Show that $\mu^*(A \cup B) + \mu^*(A \cap B) \;=\; \mu^*(A) + \mu^*(B)$.

2.22. Let μ^* be an outer measure on X. Show that if A, B are disjoint subsets of X and A is μ^*-measurable, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

2.23. Given an uncountable set X, define $\mu^*(E) = 0$ if $E \subseteq X$ is countable, and $\mu^*(E) = 1$ if $E \subseteq X$ is uncountable, Show that μ^* is an outer measure on X, and identify the μ^*-measurable subsets of X.