2.4 The Completion of a Measure

By definition, a set $E \subseteq X$ is a null set for a measure μ on X if $E \in \Sigma$ and $\mu(E) = 0$. In general, an arbitrary subset A of E need not be measurable, but if A happens to be measurable then monotonicity implies that $\mu(A) = 0$. A *complete measure* is one such that every subset A of every null set E is measurable (Definition 2.19).

Complete measures are often more convenient to work with than incomplete measures. Fortunately, if we have a incomplete measure μ in hand, there is a way to extend μ to a larger σ-algebra $\overline{\Sigma}$ in such a way that the extended measure is complete. This new extended measure $\overline{\mu}$ is called the *completion* of μ, and its construction is given in the next exercise.

Exercise 2.25. Let (X, Σ, μ) be a measure space, and let $\mathcal{N}$ be the collection of all μ-null sets in X:

$$\mathcal{N} \;=\; \bigl\{N \in \Sigma : \mu(N) = 0\bigr\}.$$

Define

$$\overline{\Sigma} \;=\; \{E \cup Z : E \in \Sigma,\; Z \subseteq N \in \mathcal{N}\},$$

and prove the following statements.

(a) $\overline{\Sigma}$ is a σ-algebra on X.

(b) For each set $E \cup Z \in \overline{\Sigma}$, define

$$\overline{\mu}(E \cup Z) \;=\; \mu(E).$$

Then $\overline{\mu}$ is a well-defined function on $\overline{\Sigma}$.

(c) $\overline{\mu}$ is a measure on $(X, \overline{\Sigma})$.

(d) $\overline{\mu}$ is the *unique* measure on $(X, \overline{\Sigma})$ that coincides with μ on Σ.

(e) $\overline{\mu}$ is complete. $\diamondsuit$

Example 2.26. Let $\mathcal{B}_{\mathbb{R}^d}$ be the Borel σ-algebra on $\mathbb{R}^d$, and let μ be Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. Since every open subset of $\mathbb{R}^d$ is Lebesgue measurable, $\mathcal{B}_{\mathbb{R}^d}$ is contained in the σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ of Lebesgue measurable subsets of $\mathbb{R}^d$. By Theorem 1.37, the σ-algebra $\overline{\mathcal{B}_{\mathbb{R}^d}}$ constructed in Exercise 2.25 is precisely $\mathcal{L}_{\mathbb{R}^d}$, and $\overline{\mu}$ is Lebesgue measure $|\cdot|$ on $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d})$. $\diamondsuit$

Example 2.27. Consider the δ-measure as a measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. In this case $\overline{\mathcal{B}_{\mathbb{R}^d}} = \mathcal{P}(\mathbb{R}^d)$, and $\overline{\delta} = \delta$ on $(\mathbb{R}^d, \mathcal{P}(\mathbb{R}^d))$. $\diamondsuit$