2.2 Measures

Now we turn to measures themselves. Although we gave the abstract definition of a measure earlier in Definition 1.1, we recall the explicit definition here.

Definition 2.8 (Measure). Let (X, Σ) be a measurable space. A function $\mu\colon \Sigma \to [0,\infty]$ is a *measure* on (X, Σ) if $\mu(\emptyset) = 0$ and μ is *countably additive*:

$$E_1, E_2, \dots \in \Sigma \text{ are disjoint} \quad \Longrightarrow \quad \mu\Bigl(\bigcup_k E_k\Bigr) \;=\; \sum_k \mu(E_k). \tag{2.3}$$

In this case we say that (X, Σ, μ) is a *measure space*. We refer to the elements of Σ as the *μ-measurable subsets* of X. If the measure μ is clear from context, then we may simply call them the *measurable subsets* of X. The number $\mu(E)$ is the *μ-measure* or simply the *measure* of $E \in \Sigma$. ♢

Remark 2.9. (a) We use the phrases "(X, Σ, μ) is a measure space" and "μ is a measure on (X, Σ)" interchangeably. If the σ-algebra Σ is understood then we often simply write "μ is a measure on X."

(b) To avoid multiplicities of parentheses, braces, and brackets, we usually write $\mu\{x\}$ instead of $\mu(\{x\})$. Similarly, if μ is a measure on $\mathbb{R}$ then we usually write $\mu[a, b)$ instead of $\mu([a, b))$, and so forth.

(c) Since $\mu(E_k) \ge 0$ for every k, the series $\sum \mu(E_k)$ that appears in equation 2.3 always exists in the sense of the extended real numbers, i.e., it either converges to a finite nonnegative number, or it is ∞.

(d) Countable additivity implies finite additivity, but Problem 2.20 shows that the converse does not hold in general. ♢

We introduce some terminology for measures with special properties.

Definition 2.10. Let (X, Σ, μ) be a measure space.

(a) If $\mu(X) < \infty$ then we say that μ is a *bounded measure* or a *finite measure*, and in this case we call (X, Σ, μ) a *finite measure space*.

(b) A measure that is not bounded is called an *unbounded measure*. In this case we say that (X, Σ, μ) is an *infinite measure space*.

(c) If there exist countably many sets $E_1, E_2, \dots \in \Sigma$ such that $\mu(E_k) < \infty$ for every k and $X = \cup E_k$, then we say that μ is a *σ-finite measure*.

(d) If every set $E \in \Sigma$ with $\mu(E) = \infty$ has a subset $F \subseteq E$ such that $F \in \Sigma$ and $\mu(F) < \infty$, then we say that μ is a *semifinite measure*. ♢

Suppose that μ is a measure on a measurable space (X, Σ), and Σ' is a σ-algebra contained in Σ. (We could call Σ' a sub-σ-algebra, but that terminology seems rather awkward, so we will usually avoid it.) In this case, $\mu|_{\Sigma'}$ (μ restricted to Σ') is a measure on (X, Σ'). Technically, μ and $\mu|_{\Sigma'}$ are two different measures, but often we refer to the restriction of μ as "μ on Σ'," or even just simply as μ. Lebesgue measure is a typical illustration of this situation, as we describe next.

Example 2.11 (Lebesgue Measure). We proved in Chapter 1 that Lebesgue measure is a measure on the measure space $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d})$. Many texts assign a symbol such as m or λ to represent Lebesgue measure, in which case we would denote the Lebesgue measure of $E \in \mathcal{L}_{\mathbb{R}^d}$ by $m(E)$ or $\lambda(E)$. However, for our purposes it is usually more convenient to simply write $|E|$ of the Lebesgue measure of E, as we did in Chapter 1, and to speak of "Lebesgue measure" without assigning a specific symbol to represent this measure.

Lebesgue measure is a σ-finite measure on $\mathbb{R}^d$ that is not a bounded measure (because $|\mathbb{R}^d| = \infty$). When we deal with Lebesgue measure in this volume, we will usually want to take the σ-algebra to be the Lebesgue σ-algebra. Therefore, when we speak of Lebesgue measure without qualification we assume that the σ-algebra is $\mathcal{L}_{\mathbb{R}^d}$. However, we sometimes need to restrict to a smaller σ-algebra. After the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$, the σ-algebra on $\mathbb{R}^d$ that we encounter most often is the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$. We refer to the restriction of Lebesgue measure to the Borel subsets of $\mathbb{R}^d$ as Lebesgue measure on $\mathcal{B}_{\mathbb{R}^d}$. $\diamondsuit$

Here are some new examples of measures.

Exercise 2.12 (The δ Measure). Let X be a set, and fix an element $a \in X$. For each set $E \subseteq X$ define

$$\delta_a(E) = \begin{cases} 1, & a \in E, \\ 0, & a \notin E. \end{cases}$$

Show that δ_a is a bounded measure with respect to the σ-algebra $\mathcal{P}(X)$ (and hence also with respect to every σ-algebra on X). $\diamondsuit$

The measure δ_a has many names, including the δ *or delta measure at* a, the *Dirac measure at* a, and the *point mass at* a. If X is a vector space and $a = 0$ then we often use the shorthand

$$\delta = \delta_0.$$

That is, δ is defined by

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

If we compare Lebesgue measure on $\mathbb{R}^d$ with a delta measure δ_a where $a \in \mathbb{R}^d$, we see many differences, such as the following.

- δ_a is defined on every subset of $\mathbb{R}^d$ (the σ-algebra is $\mathcal{P}(\mathbb{R}^d)$).
- δ_a is a bounded measure: $\delta_a(\mathbb{R}^d) = 1$.
- Singletons can have nonzero measure: $\delta_a\{a\} = 1$.
- Sets with infinite Lebesgue measure can have zero measure with respect to δ_a: $\delta_a(\mathbb{R}^d \backslash \{a\}) = 0$.
- δ_a is not translation-invariant: $\delta_a(E)$ need not equal $\delta_a(E+h)$.

Next we define a measure that is quite different from either Lebesgue measure or a delta measure.

Exercise 2.13 (Counting Measure). Let X be a set. For each set $E \subseteq X$ define

$$\mu(E) \;=\; \begin{cases} \text{cardinality of } E, & E \text{ finite}, \\ \infty, & E \text{ infinite}. \end{cases}$$

Show that μ is an unbounded measure with respect to the σ-algebra $\mathcal{P}(X)$ (and hence also with respect to every σ-algebra on X). This measure is called *counting measure* on X. $\diamondsuit$

Focusing on counting measure on $\mathbb{R}^d$, we some ways in which counting measure is similar to Lebesgue measure or a delta measure, but more ways in which it is different.

- μ is defined on every subset of $\mathbb{R}^d$ (the σ-algebra is $\mathcal{P}(\mathbb{R}^d)$).
- μ is unbounded and is not σ-finite, although it is semifinite.
- Only finite sets have finite measure with respect to μ.
- μ is translation-invariant.

Since the basic definition of σ-algebras revolves around countable unions, we can guess that it is often unpleasant to work with a measure that is not σ-finite. Fortunately, most of the measures that we encounter in practice are σ-finite.

Example 2.14. If X is an uncountable set then counting measure on X is semifinite but not σ-finite. However, in practice we usually only encounter counting measure on countable sets, such as $\mathbb{N}$ or $\mathbb{Z}^d$. Counting measure on these sets is important because we will eventually see that integration with respect to counting measure is related to to infinite series. In particular, we will see in Exercise 4.1 that if μ is counting measure on the natural numbers $\mathbb{N}$ and $f\colon \mathbb{N} \to [0,\infty]$ then the integral of f with respect to μ is given by

$$\int_{\mathbb{N}} f\,d\mu \;=\; \sum_{n=1}^{\infty} f(n).$$

Thus, results about series are just special cases of results about integration with respect to a measure. $\diamondsuit$

As we have seen, there can be many different measures on a given space. The space that we encounter most often in this volume is $\mathbb{R}^d$, and the measure that we encounter most often is Lebesgue measure. Therefore we make the following notational convention, which essentially states that Lebesgue measure will be our "default" measure on $\mathbb{R}^d$. Since Lebesgue measure is based on volumes of boxes, and volume is based on lengths determined by the Euclidean norm, this is in harmony with the usual mathematical convention that the Euclidean norm is the default norm on $\mathbb{R}^d$.

Notation 2.15 (Default Measure on $\mathbb{R}^d$). Unless we specifically state otherwise, whenever we deal with $\mathbb{R}^d$ we will implicitly assume that we have chosen to work with Lebesgue measure. Further, unless specifically stated otherwise, we will assume that the σ-algebra associated with Lebesgue measure is the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$. For example, using this convention, the phrase "let E be a measurable subset of $\mathbb{R}^d$" is interpreted to mean that E is a subset of $\mathbb{R}^d$ that is measurable with respect to Lebesgue measure (i.e., E belongs to $\mathcal{L}_{\mathbb{R}^d}$). $\diamondsuit$

Additional Problems

2.11. (a) Prove that a nonnegative finite linear combination of measures is a measure, i.e., if $\mu_1, \dots, \mu_N$ are measures on (X, Σ) and $c_1, \dots, c_N \geq 0$, then $\mu = \sum_{n=1}^N c_n \mu_n$, $E \in \Sigma$, defines a measure on (X, Σ).

(b) For each $n \in \mathbb{N}$ let μ_n be a measure on (X, Σ). What conditions on scalars c_n are needed so that $\mu(E) = \sum_{n=1}^\infty c_n \mu_n(E)$ defines a measure on (X, Σ)?

2.12. Let (X, Σ, μ) be a measure space. Given $Y \in \Sigma$, define $\mu_Y \colon \Sigma \to [0, \infty]$ by $\mu_Y(E) = \mu(E \cap Y)$. Show that μ_Y is a measure on (X, Σ).

2.13. Let μ be a measure on $\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$. Show that μ is completely determined by the values $(\mu\{k\})_{k \in \mathbb{N}}$. In other words, show that $\mu \mapsto (\mu\{k\})_{k \in \mathbb{N}}$ is an injective map of the measures on $\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$ into the space of sequences of extended real numbers. Identify the range of this map, and identify the sequences that correspond to bounded measures on $\mathbb{N}$.

2.14. (a) Given an infinite set X, for each $E \subseteq X$ define $\mu(E) = 0$ if E is finite, and $\mu(E) = \infty$ if E is infinite. Show that μ is finitely additive but is not countably additive.

(b) What if we define $\mu(\emptyset) = 0$ and $\mu(E) = \infty$ for every nonempty $E \subseteq X$?

2.15. Let X be an uncountable set. Let Σ consist of all subsets A of X such that either A or $X \backslash A$ is at most countable. Given $E \in \Sigma$, define $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ otherwise. Show that Σ is a σ-algebra and μ is a measure on (X, Σ).