2

Abstract Measure Theory

Lebesgue measure is one of the premier examples of a measure on $\mathbb{R}^d$, but it is not the only measure and certainly not the only important measure on $\mathbb{R}^d$. Further, $\mathbb{R}^d$ is not the only domain on which we encounter measures. This chapter develops the theory of measures from an abstract viewpoint. The fact that we have already examined Lebesgue measure in detail will simplify this task considerably, as there are many aspects of abstract measure theory that are precisely analogous to results for Lebesgue measure. However, other portions of the theory require some extra care. In return for developing this abstract theory of measure, we will be able in the coming chapters to construct a powerful and useful theory of integration with respect to measures.

2.1 Sigma Algebras

There are three components to any measure. First there is the set X whose subsets we wish to measure. Unfortunately, we often cannot construct a measure that is suitably well-behaved for our application on all of the subsets of X, and hence the second component of a measure is the collection Σ of subsets of X that we will actually be allowed to measure. Finally, there is the measure itself, which is a mapping $\mu \colon \Sigma \to [0, \infty]$.

We cannot choose Σ at random; it must satisfy the properties of a σ-algebra. We recall the definition here (compare Definition 1.1), and also introduce a terminology for referring to a set together with a σ-algebra.

Definition 2.1 (Sigma Algebra). Let X be a set.

(a) A *σ-algebra* or *σ-field* on X is a nonempty collection Σ of subsets of X that is closed under complements and countable unions.

(b) If Σ is a σ-algebra on X, then we call (X, Σ) a *measurable space*. $\diamondsuit$

Since a σ-algebra is closed under both complements and countable unions, it is also closed under countable intersections. Additionally, a σ-algebra is also closed under *relative complements*, for if $A,\, B \in \Sigma$ then $A \backslash B = A \cap B^{\mathrm{C}} \in \Sigma$.

If Σ is a σ-algebra then it must contain at least one subset E of X. Therefore $E^{\mathrm{C}} = X \backslash E$ belongs to Σ, and hence both $X = E \cup E^{\mathrm{C}}$ and $\emptyset = X \backslash X$ are elements of Σ. As a consequence, closure under countably infinite unions implies closure under finite unions, since

$$E_1 \cup \cdots \cup E_n \;=\; E_1 \cup \cdots \cup E_n \cup \emptyset \cup \emptyset \cup \cdots .$$

However, closure of a family under complements and *finite unions* does not imply closure under countable unions in general (see Problem 2.4).

Trivial examples of σ-algebras on a set X are $\Sigma = \{\emptyset, X\}$ and the power set $\mathcal{P}(X) = \{E : E \subseteq X\}$. By Corollary 1.39, the class $\mathcal{L}_{\mathbb{R}^d}$ of all Lebesgue measurable subsets of $\mathbb{R}^d$ is an example of a σ-algebra on $\mathbb{R}^d$.

We often encounter situations where we have a particular family $\mathcal{E}$ of subsets of X that we want to measure, but $\mathcal{E}$ is not a σ-algebra. In this case there will usually be many larger collections that are σ-algebras and include all of the sets from $\mathcal{E}$. For example, the power set $\mathcal{P}(X)$ is a σ-algebra that contains $\mathcal{E}$. However, we often seek the "smallest possible" σ-algebra that contains $\mathcal{E}$. The next exercise constructs this smallest σ algebra.

Exercise 2.2. Let $\mathcal{E}$ be a collection of subsets of a set X. Show that

$$\Sigma(\mathcal{E}) \;=\; \bigcap \bigl\{\Sigma : \Sigma \text{ is a } \sigma\text{-algebra and } \mathcal{E} \subseteq \Sigma\bigr\} \tag{2.1}$$

is a σ-algebra on X. We call $\Sigma(\mathcal{E})$ the *σ-algebra generated by* $\mathcal{E}$. $\diamondsuit$

Note that if Σ_1, Σ_2 are σ-algebras, then $\Sigma_1 \cap \Sigma_2$ is not formed by intersecting the elements of Σ_1 with those of Σ_2. That is, $\Sigma_1 \cap \Sigma_2$ does *not* mean $\{A \cap B : A \in \Sigma_1,\, B \in \Sigma_2\}$. Rather, $\Sigma_1 \cap \Sigma_2$ is the collection of all sets that are *common* to both Σ_1 and Σ_2:

$$\Sigma_1 \cap \Sigma_2 \;=\; \bigl\{A : A \in \Sigma_1 \text{ and } A \in \Sigma_2\bigr\},$$

and similar remarks apply to the intersection appearing in equation (2.1).

The following exercise explains why we often refer to $\Sigma(\mathcal{E})$ as the *smallest* σ-algebra that contains $\mathcal{E}$.

Exercise 2.3. Let $\mathcal{E}$ be a collection of subsets of a set X. Show that $\Sigma(\mathcal{E})$ contains $\mathcal{E}$, and if Σ is any other σ-algebra that contains $\mathcal{E}$ then $\Sigma(\mathcal{E}) \subseteq \Sigma$.

Here is an example of a σ-algebra and a generating family for that σ-algebra.

Exercise 2.4. Given a set X, let Σ consist of all sets $E \subseteq X$ such that at least one of E or $X \backslash E$ is countable.

(a) Show that Σ is a σ-algebra on X.

(b) Let $\mathcal{S} = \{\{x\} : x \in X\}$ be the set of all singletons of elements of X. Show that $\Sigma = \Sigma(\mathcal{S})$, i.e., Σ is the σ-algebra generated by $\mathcal{S}$.

When working with $\mathbb{R}^d$, we inevitably must deal with the topology of $\mathbb{R}^d$. By Lemma 1.20, the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ contains all of the open subsets of $\mathbb{R}^d$, but is it the *smallest* σ-algebra that contains the open sets? Problem 2.10 shows that the answer to this is no—the σ-algebra generated by the open subsets of $\mathbb{R}^d$ is a *proper* subset of $\mathcal{L}_{\mathbb{R}^d}$. We have a special name for this σ-algebra.

Definition 2.5 (Borel σ-algebra on $\mathbb{R}^d$). The *Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$ on $\mathbb{R}^d$* is the smallest σ-algebra that contains all the open subsets of $\mathbb{R}^d$. That is, if we set $\mathcal{U} = \{U \subseteq \mathbb{R}^d : U \text{ is open}\}$, then

$$\mathcal{B}_{\mathbb{R}^d} \;=\; \Sigma(\mathcal{U}).$$

The elements of $\mathcal{B}_{\mathbb{R}^d}$ are called the *Borel subsets of* $\mathbb{R}^d$. $\diamondsuit$

In particular, $\mathcal{B}_{\mathbb{R}^d}$ includes all of the open and closed subsets of $\mathbb{R}^d$, as well as the G_δ and F_σ sets that were introduced in Definition 1.34. However, not every Lebesgue measurable subset of $\mathbb{R}^d$ is a Borel set (Problem 2.10), and not every Borel set is a G_δ or an F_σ set (Problem 1.15). On the other hand, Theorem 1.37 tells us that if E is a Lebesgue measurable subset of $\mathbb{R}^d$ then there exists a G_δ set H that contains E and satisfies $|H \backslash E| = 0$. Hence we can obtain the Lebesgue σ-algebra by “adjoining” sets of measure zero to the Borel σ-algebra: A set E belongs to $\mathcal{L}_{\mathbb{R}^d}$ if and only if can be written as $E = H \cup Z$ where $H \in \mathcal{B}_{\mathbb{R}^d}$ and $|Z| = 0$.

The Borel subsets of $\mathbb{R}^d$ are generated from the open subsets of $\mathbb{R}^d$. There is nothing special about $\mathbb{R}^d$ in this regard—whenever we have a space that has a topology, we can define Borel sets in a similar manner. This is stated precisely in the following definition.

Definition 2.6 (Borel σ-algebra on X). Given a topological space X, the *Borel σ-algebra $\mathcal{B}_X$ on X* is the smallest σ-algebra that contains all of the open subsets of X. The elements of $\mathcal{B}_X$ are called the *Borel subsets of X*. $\diamondsuit$

We end this section by pointing out a very simple but extremely useful fact about countable unions of sets.

Exercise 2.7 (The Disjointization Trick). If Σ is a σ-algebra of subsets of X and $E_1, E_2, \dots \in \Sigma$, then the sets F_k defined by

$$F_1 \;=\; E_1, \quad F_2 \;=\; E_2 \backslash E_1, \quad F_3 \;=\; E_3 \backslash (E_1 \cup E_2), \quad \dots \tag{2.2}$$

are disjoint, belong to Σ, and satisfy

$$\bigcup_k F_k \;=\; \bigcup_k E_k. \qquad \diamondsuit$$

This disjointization trick can be used to give some equivalent characterizations of σ-algebras (see Problem 2.4).

Additional Problems

2.1. Suppose that $\mathcal{E}$, $\mathcal{F} \subseteq \mathcal{P}(X)$. Show that if $\mathcal{E} \subseteq \Sigma(\mathcal{F})$ then $\Sigma(\mathcal{E}) \subseteq \Sigma(\mathcal{F})$.

2.2. Let (X, Σ) be a measure space. Given $Y \subseteq X$, set $\Sigma_Y = \{E \cap Y : E \in \Sigma\}$. Show that Σ_Y is a σ-algebra on Y.

2.3. Show that $\Sigma = \{E \subseteq \mathbb{R}^d : |E| = 0 \text{ or } |\mathbb{R}^d \backslash E| = 0\}$ is a σ-algebra on $\mathbb{R}^d$.

2.4. An *algebra* (or *field*) is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under complements and *finite* unions.

(a) Show that an algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable disjoint unions.

(b) Show that an algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions.

2.5. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbb{R}$. Show that each of the following collections generates $\mathcal{B}_{\mathbb{R}}$.

(a) $\mathcal{E}_1 = \{(a, b) : a < b\}$.
(b) $\mathcal{E}_2 = \{[a, b] : a < b\}$.
(c) $\mathcal{E}_3 = \{[a, b) : a < b\}$.
(d) $\mathcal{E}_4 = \{(a, \infty) : a \in \mathbb{R}\}$.
(e) $\mathcal{E}_5 = \{[a, \infty) : a \in \mathbb{R}\}$.
(f) $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$.
(g) $\mathcal{E}_7 = \{(-\infty, a] : a \in \mathbb{R}\}$.
(h) $\mathcal{E}_8 = \{(r, \infty) : r \in \mathbb{Q}\}$.
(i) $\mathcal{E}_9 = \{(-\infty, r) : r \in \mathbb{Q}\}$.

2.6. Let $\mathcal{B}_{\mathbb{R}^2}$ be the Borel σ-algebra on $\mathbb{R}^2$. Show that each of the following collections generates $\mathcal{B}_{\mathbb{R}^2}$.

(a) $\mathcal{E}_1 = \{(a, b) \times (c, d) : a < b, c < d\}$.
(b) $\mathcal{E}_2 = \{(a, b) \times \mathbb{R} : a < b\} \cup \{\mathbb{R} \times (c, d) : c < d\}$.

2.7. Show that any σ-algebra Σ that contains infinitely many subsets of X must be uncountable.

2.8. Given a subset A of a set X, show that

$$\Sigma = \{S : S \subseteq A\} \cup \{X \backslash S : S \subseteq A\}$$

is a σ-algebra on X.

2.9. Suppose that $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ are nested increasing σ-algebras on X. Must $\Sigma = \cup \Sigma_n$ be a σ-algebra?

2.10. Let $A = g^{-1}(N)$ be the set discussed in Problem 1.36(c). Show that although A is Lebesgue measurable, it is not a Borel set.

Remark 1: Observe that if we set $h = g^{-1}$, then $h\colon [0,2] \to [0,1]$ is continuous and $h^{-1}(A) = N$. Therefore the inverse image of a Lebesgue measurable set under a continuous function need not be Lebesgue measurable.

Remark 2: The existence of Lebesgue measurable sets that are not Borel sets can also be proved by a cardinality argument. If we let $\mathcal{G}$ be the set of all open subsets of $\mathbb{R}$, then $|\mathcal{G}| = \mathfrak{c}$, the cardinality of the real line. An argument based on transfinite induction can be used to show that the cardinality of $\mathcal{B}_{\mathbb{R}}$ is also $\mathfrak{c}$. However, the Cantor set has cardinality $\mathfrak{c}$ and every subset of C is measurable, so the cardinality of the set of Lebesgue measurable subsets of $\mathbb{R}$ is at least the cardinality of the power set of C, which is strictly greater than $\mathfrak{c}$ (see [Fol99, Sec. 1.6]).