1.2 Lebesgue Measure

In Section 1.1 we defined the exterior Lebesgue measure of every subset of $\mathbb{R}^d$. Unfortunately, a major disadvantage of exterior measure is that it does not satisfy countable additivity. (we will prove this in Section 1.7). In general, the exterior measure of a countable (even finite) union of disjoint sets need not equal the sum of the exterior measures of the sets! In this section we will construct a σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ of subsets of $\mathbb{R}^d$ such that countable additivity of exterior measure holds *when we restrict* to sets in $\mathcal{L}_{\mathbb{R}^d}$.

To motivate the definition of $\mathcal{L}_{\mathbb{R}^d}$, recall from Theorem 1.17 that every set $E \subseteq \mathbb{R}^d$ can be surrounded by an open set U that satisfies

$$|E|_e \;\le\; |U|_e \;\le\; |E|_e + \varepsilon. \tag{1.5}$$

Note that

$$U \;=\; E \cup (U \backslash E),$$

and the sets E and $U \backslash E$ are disjoint. By subadditivity,

$$|U|_e \;=\; \bigl|E \cup (U \backslash E)\bigr|_e \;\le\; |E|_e + |U \backslash E|_e. \tag{1.6}$$

Comparing equations (1.5) and (1.6), we are very tempted to infer that $|U \backslash E|_e$ must be small. However, by themselves these equations do not imply that

$$|U \backslash E|_e \;\le\; \varepsilon \qquad (\leftarrow \text{WE DO NOT KNOW THIS!}).$$

On the other hand, *if* we knew that

$$|U|_e \;=\; |E|_e + |U \backslash E|_e \qquad (\leftarrow \text{WE DO NOT KNOW THIS EITHER!}),$$

then we would have enough information to conclude that $|U \backslash E|_e \le \varepsilon$. Unfortunately, we just don't have that information, and in fact we will see examples later where this is *false*. There exists a bounded set $N \subseteq \mathbb{R}^d$ such that for some $\varepsilon > 0$, every open set $U \supseteq N$ satisfies

$$|U \backslash N|_e \;>\; \varepsilon,$$

even though there is at least one open set $U \supseteq N$ that satisfies

$$|N|_e \;\le\; |U|_e \;\le\; |N|_e + \varepsilon$$

(see Theorem 1.49 and Example 1.50). These sets N are very strange indeed. Perhaps if we ignore them they won't bother us too much. Let's pretend they don't exist—or, more precisely, let us restrict our attention to sets that do *not* exhibit this bizarre behavior.

Definition 1.19. A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable*, or simply *measurable* for short, if

$$\forall\, \varepsilon > 0, \quad \exists \text{ open } U \supseteq E \text{ such that } |U \backslash E|_e \le \varepsilon.$$

If E is Lebesgue measurable, then its *Lebesgue measure* is $|E| = |E|_e$. We denote the family of measurable sets by

$$\mathcal{L}_{\mathbb{R}^d} \;=\; \bigl\{E \subseteq \mathbb{R}^d : E \text{ is Lebesgue measurable}\bigr\},$$

and refer to $\mathcal{L}_{\mathbb{R}^d}$ as the *Lebesgue σ-algebra on* $\mathbb{R}^d$ (though we have not yet proved that it is a σ-algebra!). $\diamondsuit$

There is no difference between the numeric value of the Lebesgue measure and the exterior Lebesgue measure of a measurable set, but when we know that a set E is measurable we denote this value using the symbols $|E|$ instead of $|E|_e$.

Here are some examples of measurable sets.

Lemma 1.20. *Every open set $U \subseteq \mathbb{R}^d$ is Lebesgue measurable.*

Proof. This follows from the fact that $U \supseteq U$ and $|U \backslash U|_e = 0 < \varepsilon$. $\square$

Lemma 1.21. *If $Z \subseteq \mathbb{R}^d$ and $|Z|_e = 0$ then Z is Lebesgue measurable.*

Proof. Suppose that $|Z|_e = 0$. If we fix $\varepsilon > 0$, then by Theorem 1.17 there exists an open set $U \supseteq Z$ such that

$$|U|_e \;\le\; |Z|_e + \varepsilon \;=\; 0 + \varepsilon \;=\; \varepsilon.$$

Since $U \backslash Z \subseteq U$, it follows from monotonicity that

$$|U \backslash Z|_e \;\le\; |U|_e \;\le\; \varepsilon.$$

Hence Z is measurable. $\square$

By monotonicity, if $|Z|_e = 0$ and $E \subseteq Z$, then $|E|_e = 0$. Therefore *every subset of a zero measure set is Lebesgue measurable*. In the language of abstract measure theory, the measure space $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d}, |\cdot|)$ is therefore said to be *complete* (see Definition 2.17).

Our main goal for this section is to show that the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ really is a σ-algebra of subsets of $\mathbb{R}^d$ in the sense of Definition 1.1(a). To do this, we must show that $\mathcal{L}_{\mathbb{R}^d}$ is closed under countable unions and under complements. We begin with countable unions.

Theorem 1.22 (Closure Under Countable Unions). *If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$ are Lebesgue measurable sets then $E = \cup E_k$ is measurable, and*

$$|E| \;\le\; \sum_{k=1}^{\infty} |E_k|.$$

Proof. Fix $\varepsilon > 0$. Since E_k is measurable, there exists an open set $U_k \supseteq E_k$ such that

$$|U_k \backslash E_k|_e \le \frac{\varepsilon}{2^k}.$$

Then $U = \cup U_k$ is an open set, $U \supseteq E$, and

$$U \backslash E = \Bigl(\bigcup_k U_k\Bigr) \setminus \Bigl(\bigcup_k E_k\Bigr) \subseteq \bigcup_k (U_k \backslash E_k).$$

Hence

$$|U \backslash E|_e \le \sum_{k=1}^{\infty} |U_k \backslash E_k|_e \le \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon,$$

so E is measurable. Since we already know that exterior measure is countably subadditive, we have $|E| \le \sum |E_k|$. □

Since the empty set is measurable and has zero measure, Theorem 1.22 applies to finite as well as countably infinite unions. However, $\mathcal{L}_{\mathbb{R}^d}$ is not closed with respect to *uncountable* unions. For, if N is a set that does not belong to $\mathcal{L}_{\mathbb{R}^d}$ then Example 1.6(a) implies that N must be uncountable, yet we can write

$$N = \bigcup_{x \in N} \{x\},$$

and each singleton $\{x\}$ belongs to $\mathcal{L}_{\mathbb{R}^d}$ since it has zero measure.

As an application of the preceding results, we show next that all boxes in $\mathbb{R}^d$ are measurable.

Corollary 1.23. *Every box Q in $\mathbb{R}^d$ is Lebesgue measurable and satisfies $|\partial Q|_e = 0$ and $|Q| = |Q^\circ| = \mathrm{vol}(Q)$.*

Proof. The interior of Q is an open set, so it is measurable by Lemma 1.20. We saw in Example 1.6(b) that the boundary of Q has exterior measure zero, so Lemma 1.21 implies that ∂Q is measurable. Therefore $Q = Q^\circ \cup \partial Q$ is measurable by Theorem 1.22. The equalities $|Q| = |Q^\circ| = \mathrm{vol}(Q)$ then follow from Problem 1.2. □

Since we have shown that $\mathcal{L}_{\mathbb{R}^d}$ is closed under countable unions, to complete the proof that it is a σ-algebra we need to show that $\mathcal{L}_{\mathbb{R}^d}$ is closed under complements. Surprisingly, this is more difficult than we might expect, so we break the proof into a series of smaller results. We begin with what may seem to be an unrelated result, namely that Lebesgue measure is finitely additive for disjoint boxes. Ultimately we want to show that Lebesgue measure is countably additive on any collection of disjoint measurable sets, so this is a step both towards showing that $\mathcal{L}_{\mathbb{R}^d}$ is closed under complements and that Lebesgue measure is countably additive.

Our next lemma will show that the Lebesgue measure of a union of finitely many disjoint boxes is the sum of the measures of those boxes. In fact, since we are dealing with *boxes*, we can relax the disjointness requirement a little—we know that the boundary of a box has measure zero, so it doesn't matter if the boundaries of two boxes intersect, as long as their interiors are disjoint. We use the following terminology

Notation 1.24. We say that a collection of boxes $\{Q_k\}$ is *nonoverlapping* if their interiors Q_k° are disjoint, i.e., if $Q_j^\circ \cap Q_k^\circ = \emptyset$ whenever $j \neq k$. $\diamondsuit$

Lemma 1.25. *If $\{Q_k\}_{k=1}^N$ is a finite collection of nonoverlapping boxes, then*

$$\left|\bigcup_{k=1}^N Q_k\right| = \sum_{k=1}^N |Q_k|.$$

Proof. It follows from Theorem 1.22 that the set $E = Q_1 \cup \cdots \cup Q_N$ is measurable since each individual box is measurable. By subadditivity,

$$|E| = \left|\bigcup_{k=1}^N Q_k\right| \le \sum_{k=1}^N |Q_k|,$$

so our task is to prove the opposite inequality.

Let $\{R_\ell\}$ be a cover of $Q_1 \cup \cdots \cup Q_N$ by countably many boxes. For each fixed k, the collection $\{R_\ell \cap Q_k\}_\ell$ is a covering of Q_k by boxes, so

$$|Q_k| \le \sum_\ell |R_\ell \cap Q_k|, \qquad k = 1, \dots, N.$$

Also, $\{R_\ell \cap Q_k\}_{k=1}^N$ is a finite collection of nonoverlapping boxes contained in R_ℓ, so a computation of volumes of boxes implies that

$$\sum_{k=1}^N |R_\ell \cap Q_k| \le |R_\ell|.$$

Therefore

$$\sum_{k=1}^N |Q_k| \le \sum_{k=1}^N \sum_\ell |R_\ell \cap Q_k| \le \sum_\ell |R_\ell|.$$

Since this is true for every covering, we conclude that

$$\sum_{k=1}^N |Q_k| \le \inf\Big\{\sum_\ell |R_\ell|\Big\} = \left|\bigcup_{k=1}^N Q_k\right|,$$

where the infimum is taken over all possible coverings of $Q_1 \cup \cdots \cup Q_N$ by countably many boxes R_ℓ. $\square$

Our next goal is to show that additivity of exterior Lebesgue measure holds for any two sets that are separated by a positive distance, where distance is defined as follows.

Definition 1.26. The *distance* between two sets A, $B \subseteq \mathbb{R}^d$ is

$$\operatorname{dist}(A, B) \;=\; \inf\bigl\{|x-y| : x \in A,\, y \in B\bigr\}. \qquad \diamondsuit$$

We prove now that additivity of exterior measure holds for any two arbitrary sets that are a positive distance apart. That is, given any sets A and B in $\mathbb{R}^d$, even possibly nonmeasurable, we will show that if the distance between A and B is strictly positive then the exterior measure of their union is the sum of of their exterior measures.

Lemma 1.27. *If* A, $B \subseteq \mathbb{R}^d$ *satisfy* $\operatorname{dist}(A,B) > 0$, *then*

$$|A \cup B|_e \;=\; |A|_e + |B|_e.$$

Proof. By subadditivity,

$$|A \cup B|_e \;\le\; |A|_e + |B|_e,$$

so we must prove the opposite inequality.

Fix $\varepsilon > 0$. By definition of exterior measure, there exist countably many boxes Q_k such that $A \cup B \subseteq \cup Q_k$ and

$$\sum_k |Q_k| \;\le\; |A \cup B|_e + \varepsilon.$$

By dividing each Q_k into finitely many subboxes if necessary, we can assume that the diameter of each Q_k is less than $\operatorname{dist}(A, B)$:

$$\operatorname{diam}(Q_k) \;=\; \sup\bigl\{|x-y| : x, y \in Q_k\bigr\} \;<\; \operatorname{dist}(A,B).$$

Consequently, each box Q_k can intersect at most one of A or B. Therefore we can divide our sequence of boxes into three distinct subsequences. First, we let $\bigl\{Q_k^A\bigr\}$ be the subsequence of $\{Q_k\}$ that contains those boxes that intersect A. Second, $\bigl\{Q_k^B\bigr\}$ is subsequence of boxes that intersect B, and finally $\bigl\{Q_k^\emptyset\bigr\}$ is subsequence of boxes that intersect neither A nor B. Since $\{Q_k\}$ covers $A \cup B$, we must have

$$A \;\subseteq\; \bigcup_k Q_k^A \qquad \text{and} \qquad B \;\subseteq\; \bigcup_k Q_k^B.$$

As $\bigl\{Q_k^A\bigr\} \cup \bigl\{Q_k^B\bigr\}$ is a subcollection of $\{Q_k\}$, we conclude that

$$|A|_e + |B|_e \;\le\; \sum_k |Q_k^A| \;+\; \sum_k |Q_k^B| \;\le\; \sum_k |Q_k| \;\le\; |A \cup B|_e + \varepsilon.$$

Since ε is arbitrary so this tells us that $|A|_e + |B|_e \le |A \cup B|_e$. $\square$

The next exercise applies Lemma 1.27 to compact subsets of $\mathbb{R}^d$.

Exercise 1.28. (a) Prove that if A, B are nonempty, compact, disjoint subsets of $\mathbb{R}^d$ then $\operatorname{dist}(A,B) > 0$. Even so, show there exist nonempty, closed, disjoint sets A, B that satisfy $\operatorname{dist}(A,B) = 0$.

(b) Given finitely many disjoint compact set $F_1, \dots, F_N \subseteq \mathbb{R}^d$, show that

$$\left| \bigcup_{k=1}^{N} F_k \right|_e = \sum_{k=1}^{N} |F_k|_e. \qquad \diamondsuit \tag{1.7}$$

Although we know that all open sets are measurable, we have not yet proved that $\mathcal{L}_{\mathbb{R}^d}$ is closed under complements, so we do not yet know whether compact sets are measurable. This is why we stated equation (1.7) in terms of exterior Lebesgue measure $|\cdot|_e$ rather than Lebesgue measure $|\cdot|$. However, our very next result will give a direct proof that all compact subsets of $\mathbb{R}^d$ are measurable.

Theorem 1.29. *Every compact set $F \subseteq \mathbb{R}^d$ is Lebesgue measurable.*

Proof. Since F is bounded, it has finite exterior measure. Fix any $\varepsilon > 0$. Then there exists an open set $U \supseteq F$ such that

$$|U| \le |F|_e + \varepsilon.$$

Our task is to show that $|U \backslash F|_e \le \varepsilon$.

Appealing to Problem 1.11, since $U \backslash F$ is open there exist countably many nonoverlapping boxes $\{Q_k\}_{k \in \mathbb{N}}$ such that

$$U \backslash F = \bigcup_{k=1}^{\infty} Q_k.$$

For any finite N, the set

$$R_N = \bigcup_{k=1}^{N} Q_k$$

is compact, and Theorem 1.22 implies that it is measurable since each box Q_k is measurable. Since the Q_k are nonoverlapping boxes, Lemma 1.25 therefore tells us that

$$|R_N| = \sum_{k=1}^{N} |Q_k|.$$

Now, $R_N \subseteq U \backslash F$, so R_N and F are disjoint compact sets that are both contained in U. Applying Exercise 1.28(b) and monotonicity, we see that

$$\begin{aligned} |F|_e + \sum_{k=1}^{N} |Q_k| &= |F|_e + |R_N| \\ &= |F \cup R_N|_e \\ &\le |U| \\ &\le |F|_e + \varepsilon. \end{aligned}$$

Since all of the quantities appearing on the preceding line are finite, we obtain

$$\sum_{k=1}^{N} |Q_k| \leq \varepsilon.$$

Taking the limit as $N \to \infty$, it follows that

$$|U \backslash F|_e = \left| \bigcup_{k=1}^{\infty} Q_k \right| \leq \sum_{k=1}^{\infty} |Q_k| = \lim_{N\to\infty} \sum_{k=1}^{N} |Q_k| \leq \varepsilon. \qquad \square$$

Now that we know that compact sets are measurable, we will use closure under countable unions to prove that arbitrary closed sets are measurable.

Corollary 1.30. *Every closed set $F \subseteq \mathbb{R}^d$ is Lebesgue measurable.*

Proof. Let $B_k = \{x \in \mathbb{R}^d : |x| \leq k\}$ be the closed ball in $\mathbb{R}^d$ of radius k centered at the origin. Then $F_k = F \cap B_k$ is compact and hence is measurable. Since F is the union of the countably many sets F_k, $k \in \mathbb{N}$, we conclude that F is measurable. $\square$

Remark 1.31. Note that the proof of Corollary 1.30 relies on the fact that we can write $\mathbb{R}^d$ as a countable union of sets that each have finite Lebesgue measure. Because there exist measurable sets $E_k \subseteq \mathbb{R}^d$ with $|E_k| < \infty$ such that $\mathbb{R}^d = \cup_{k=1}^{\infty} E_k$, we say that Lebesgue measure is *σ-finite* (compare the definition of σ-finiteness for abstract measures that is given in Definition 2.9). $\diamondsuit$

At last, we come to the proof that $\mathcal{L}_{\mathbb{R}^d}$ is closed under complements.

Theorem 1.32 (Closure Under Complements). *If $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, then so is $E^{\mathrm{C}} = \mathbb{R}^d \backslash E$.*

Proof. Suppose that E is measurable. Then for each k we can find an open set $U_k \supseteq E$ such that $|U_k \backslash E|_e < \frac{1}{k}$. Define

$$F_k = U_k^{\mathrm{C}}.$$

Then F_k is closed and hence is measurable. Let

$$H = \bigcup_k F_k = \bigcup_k U_k^{\mathrm{C}}.$$

Then H is measurable and $H \subseteq E^{\mathrm{C}}$. Let $Z = E^{\mathrm{C}} \backslash H$. Then for any fixed j we have

$$Z = E^{\mathrm{C}} \setminus \bigcup_k U_k^{\mathrm{C}} \subseteq E^{\mathrm{C}} \backslash U_j^{\mathrm{C}} = U_j \backslash E.$$

Hence

$$|Z|_e \leq |U_j \backslash E|_e < \frac{1}{j}.$$

This is true for every $j \in \mathbb{N}$, so $|Z|_e = 0$. But then Z is measurable, so $E^{\mathrm{C}} = H \cup Z$ is measurable as well. $\square$

Thus $\mathcal{L}_{\mathbb{R}^d}$ is closed under complements and countable unions. In the terminology of Definition 1.1, we have shown that $\mathcal{L}_{\mathbb{R}^d}$ is a σ-algebra on $\mathbb{R}^d$.

Corollary 1.33. *The collection $\mathcal{L}_{\mathbb{R}^d}$ of Lebesgue measurable subsets of $\mathbb{R}^d$ is a σ-algebra on $\mathbb{R}^d$.* $\diamondsuit$

It is useful to observe that since the complement of a union is an intersection, $\mathcal{L}_{\mathbb{R}^d}$ is closed under countable intersections as well as countable unions.

Additional Problems

1.11. Show that if U is an open subset of $\mathbb{R}^d$, then there exist countably many nonoverlapping boxes $\{Q_k\}_{k\in\mathbb{N}}$ such that $U = \cup Q_k$.

1.12. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *Lipschitz* if there exists a constant $C > 0$ such that $|f(x) - f(y)| \le C\,|x - y|$ for all $x,\, y \in \mathbb{R}$.

(a) Show that if f is differentiable at every point and f' is bounded, then f is Lipschitz. Give an example of a Lipschitz function that is not differentiable at every point.

(b) Show that a Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$ sends sets of measure zero to sets of measure zero. That is, if $Z \subseteq \mathbb{R}$ and $|Z| = 0$, then $|f(Z)| = 0$.

Remark: Lipschitz functions on $\mathbb{R}^d$ are studied in more detail in Section 1.8.