9. Open & Closed Sets

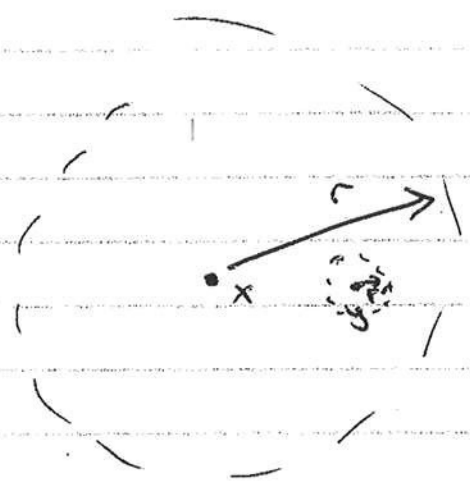
<u>Definition</u>

$G \subseteq \mathbb{R}^p$ is <u>open</u> if $\forall x \in G \ \exists r > 0$ s.t. $B_r(x) \subseteq G$.

$F \subseteq \mathbb{R}^p$ is <u>closed</u> if $\mathcal{C}(F) = \mathbb{R}^p \setminus F$ is open.

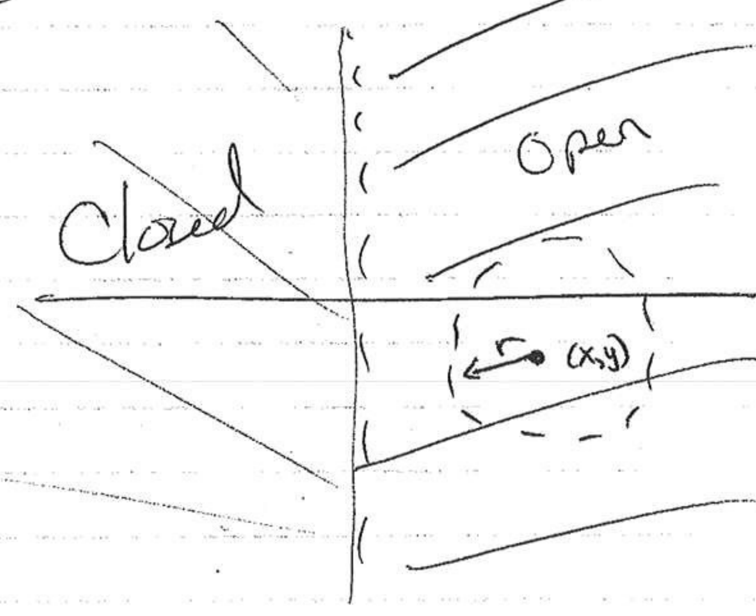
<u>Ex.</u> $\mathbb{R}^2$

$B_r(x)$ is open.



$\{(x,y) : x > 0\}$ is open

$\{(x,y) : x \leq 0\}$ is closed



$\emptyset$, $\mathbb{R}^2$ are open. <u>Exercise:</u> Cantor set F is closed.

any collection ~~[illegible]~~

(even uncountably many)

Theorem

(a) The union of open sets is open.

(b) The intersection of finitely many open sets is open.

Proof

(b) Suppose $G_1, \dots, G_n$ are open.

Suppose $x \in \bigcap_{k=1}^{n} G_k$. Then $x \in G_k \ \forall k$.

$\exists r_k$ st. $B_{r_k}(x) \subseteq G_k$.

Let $r = \min\{r_1, \dots, r_n\}$. Then $B_r(x) \subseteq G_k \ \forall k$.

So $B_r(x) \subseteq \bigcap_{k=1}^{n} G_k$. So $\bigcap_{k=1}^{n} G_k$ is open.

(a) Exercise. ∎

Ex: $\bigcap_{n=1}^{\infty} (-1/n, 1+1/n) = [0,1]$ is not closed

Exercise

a. The union of finitely many closed sets is closed.

b. The intersection of any collection of closed sets is closed (even uncountably many).

Example

The union of infinitely many closed sets need not be closed! E.g.,

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0,1)$$

Definition

If $x \in \mathbb{R}^p$, then a neighborhood of x is any set N that contains an open set that contains x:

$$x \in U \subseteq N$$

Note: A neighborhood need not itself be an open set, e.g.,

N is a neighborhood of x.

However, since a neighborhood has to contain an open set that contains x, usually any definition or theorem that is stated in terms of neighborhoods has an equivalent formulation in terms of open sets.

Further, since an open set has to contain an open ball, in the end things usually come down to questions about balls. We'll give examples of this as we go along, here is our first one.

Definition

Let $A \subseteq \mathbb{R}^p$ be given. Then $x \in \mathbb{R}^p$ is an

interior point of A if

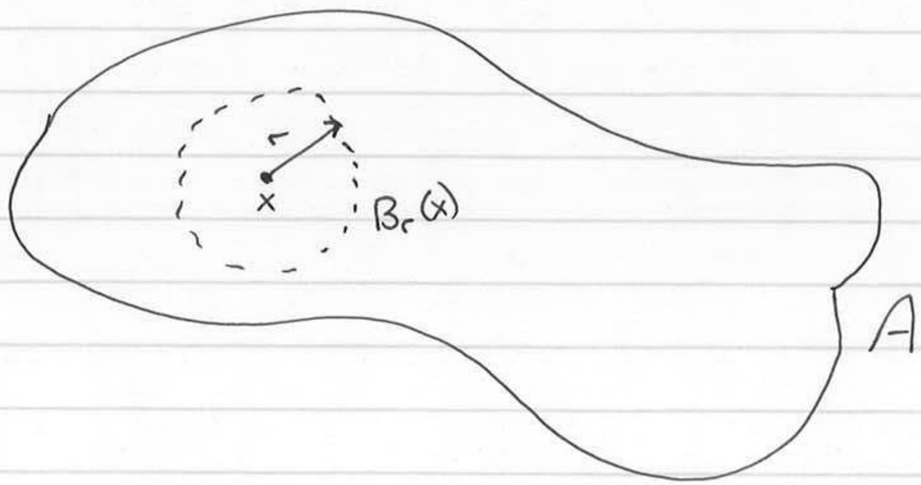
$$\exists \text{ neighborhood } N \text{ of } x \text{ s.t. } x \in N \subseteq A.$$

Exercise

Given $A \subseteq \mathbb{R}^p$, show that TFAE:

a. x is an interior point of A.
b. $\exists$ open set U s.t. $x \in U \subseteq A$.
c. $\exists\, r > 0$ s.t. $B_r(x) \subseteq A$

Thus x is an interior point if there's an open ball centered at x contained in A.



<u>Note</u>

Beware of pictures – the set A can be anything; it doesn't have to be closed, connected, or anything else. It's just easiest to draw A that way in illustrations.

<u>Exercise</u>

Given $G \subseteq \mathbb{R}^p$, show that

G is open $\Longleftrightarrow$ every $x \in G$ is an interior point of G

<u>Definition</u>

Given $A \subseteq \mathbb{R}^p$, a point $x \in \mathbb{R}^p$ is an <u>exterior point</u> to A if it is interior to $\mathcal{C}(A)$, the complement of A.

<u>Exercise</u>

Show TFAE.

a. x is an exterior point to A.

b. $\exists$ neighborhood N of x s.t. $N \subseteq \mathcal{C}(A)$

c. $\exists$ open set U s.t. $x \in U \subseteq \mathcal{C}(A)$

d. $\exists\ r > 0$ s.t. $B_r(x) \subseteq \mathcal{C}(A)$.

<u>Note</u>

If x is an interior point to A then x must belong to A (why?), & if x is an exterior point, then x must belong to $\mathcal{C}(A)$ (why?)

Notation

We write

$$A^{\circ} = \{x \in A : x \text{ is an interior point of } A\}$$

We call A° the interior of A.

Likewise, the exterior of A is $\mathcal{C}(A)^{\circ}$, the interior of the complement of A.

Exercise

Given $G \subseteq \mathbb{R}^p$, show

$$G \text{ is open} \iff G = G^{\circ}.$$

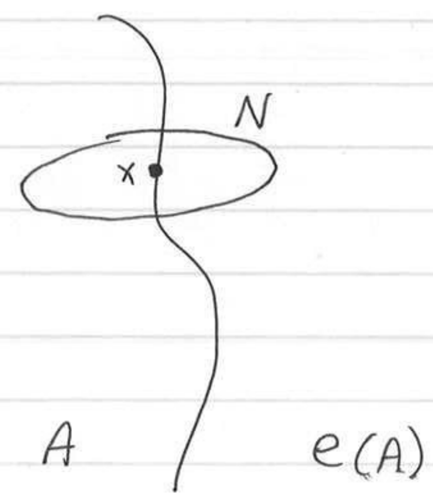
Now we come to something a little trickier: boundary points. A boundary point is neither an interior nor an exterior point, which yields the following precise definition.

Definition

Let $A \subseteq \mathbb{R}^p$ be given. Then $x \in \mathbb{R}^p$ is a boundary point of A if

$$\forall \text{ neighborhood } N \text{ of } x, \quad N \cap A \neq \emptyset \;\; \& \;\; N \cap \mathcal{C}(A) \neq \emptyset.$$

That is, every neighborhood of x must contain both points of A & points of $\mathcal{C}(A)$



We define

$$\partial A = \{x \in \mathbb{R}^p : x \text{ is a boundary point of } A\}$$

and we call ∂A the <u>boundary of A</u>.

<u>Exercise</u>

Give an example of a set A s.t. $\partial A \subseteq A$.

" " " " " " " " $\partial A \subseteq \mathcal{C}(A)$

" " " " " " " " ∂A is not contained in either A or $\mathcal{C}(A)$.

<u>Exercise</u>

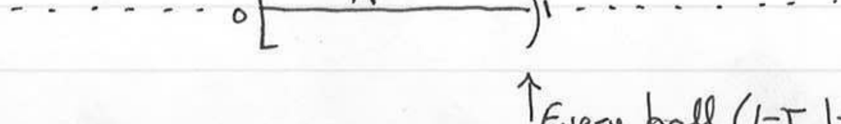
Let $A \subseteq \mathbb{R}^p$ be given. Prove TFAE.

a. x is a boundary point of A, i.e., $x \in \partial A$.

b. $\forall$ open set U containing x, $A \cap U \neq \emptyset$ & $\mathcal{C}(A) \cap U \neq \emptyset$

c. $\forall r > 0$, $A \cap B_r(x) \neq \emptyset$ & $\mathcal{C}(A) \cap B_r(x) \neq \emptyset$

<u>Example</u> If $A = [0,1)$ then $\partial A = \{0,1\}$.



$\uparrow$ Every ball $(1-r, 1+r)$ centered at $x=1$ contains points of A & $\mathcal{C}(A)$.

In fact, we don't even need to consider all possible balls - it's enough to consider those of radii $\frac{1}{n}$. (there's nothing special about $1/n$, we could use $1/2^n$, etc., instead)

Theorem

Given $A \subseteq \mathbb{R}^p$ & $x \in \mathbb{R}^p$, TFAE:

a. x is a boundary point of A.

b. $\forall n \in \mathbb{N}$, $\exists a_n \in A$, $\exists b_n \in \mathcal{C}(A)$ s.t.

$$\|x - a_n\| < \frac{1}{n} \quad \& \quad \|x - b_n\| < \frac{1}{n}.$$

Proof

$a \Rightarrow b$. Suppose a is a boundary point of A.

Fix n, and let $r = \frac{1}{n}$. By the preceding exercise, $A \cap B_r(x) \neq \emptyset$, so $\exists\, a_n \in A \cap B_r(x)$.

By definition, $a_n \in A$ and $\|x - a_n\| < r = \frac{1}{n}$.

Similarly, $\mathcal{C}(A) \cap B_r(x) \neq \emptyset$ so $\exists\, b_n \in \mathcal{C}(A) \cap B_r(x)$.

$b. \Rightarrow a.$ Exercise. $\blacksquare$

Exercise $G \subseteq \mathbb{R}^p$ is open $\iff$ every $x \in G$ is an interior point

Theorem $F \subseteq \mathbb{R}^p$ is closed $\iff$ F contains all its boundary points

Proof:

$\Rightarrow$ Assume F is closed, $x \in \partial F$.

If $x \notin F$ then $x \in \mathcal{C}(F)$, which is open.

So $\exists\, r > 0$ s.t. $B_r(x) \subseteq \mathcal{C}(F)$. But then $x \notin \partial F$.

$\Leftarrow$ Assume F contains all boundary pts.

Assume $y \in \mathcal{C}(F)$. Then y is not a boundary pt.

Further, y is not an interior point. Hence

y is an exterior point, so $\exists\, r$ s.t. $B_r(y) \subseteq \mathcal{C}(F)$.

Hence $\mathcal{C}(F)$ is open, so F is closed. $\blacksquare$

Note

Every open subset of $\mathbb{R}^p$ can be written as a union of open balls, because if $G \subseteq \mathbb{R}^p$ is open, then for every $x \in G$ there exists an r_x s.t. $B_{r_x}(x) \subseteq G$, & hence

$$G = \bigcup_{x \in G} B_{r_x}(x) \qquad \text{(why does equality hold?)}$$

In general, however, there will be uncountably many points in G (exercise: what is the only open set containing only countably many points?).

Is it possible to write G as a union of countably many balls? We certainly can't hope to get away with only finitely many balls, e.g., consider the open unit square or cube:

This set <u>isn't</u> a union of <u>finitely</u> many balls, and it <u>is</u> a union of <u>uncountably</u> many balls – can we get away with using <u>countably</u> many balls?

First guess: Just take balls centered at points that have <u>rational</u> coordinates. That is, let

$$R = \{ q \in G : q = (q_1, \ldots, q_p) \text{ with } q_i \in \mathbb{Q} \}.$$

This set is countable (why?), and since G is open, for each $q \in R$ there exists some $r_q > 0$ s.t. $B_{r_q}(q) \subseteq G$. Therefore, we will certainly have

$$\underbrace{\bigcup_{q \in R} B_{r_q}(q)}_{\substack{\text{union of countably} \\ \text{many balls}}} \subseteq G \qquad (*)$$

but are we certain that equality will hold in $(*)$?

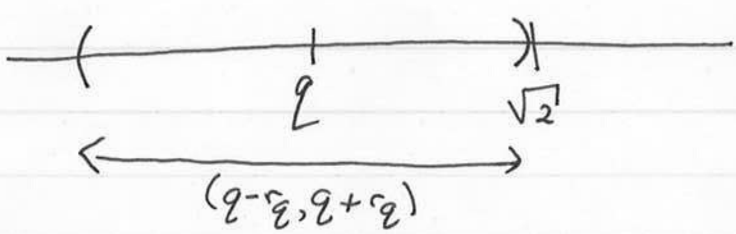
In general, NO!

Example

Consider $G = \mathbb{R}$. Then $R = \mathbb{Q}$. For each q, let $r_q = |q - \sqrt{2}|$, the distance from q to $\sqrt{2}$. Then

$$B_{r_q}(q) = (q - r_q, q + r_q)$$

does not contain $\sqrt{2}$.



Hence
$$\sqrt{2} \notin \bigcup_{q \in \mathbb{Q}} B_{\tilde{q}}(q).$$

We've put all ball over <u>every</u> rational point and yet we didn't cover the whole line!

Moral: We must be very careful about how we choose the radii!

Theorem

If $G \subseteq \mathbb{R}^p$ is open, then we can write as the union of countably many open balls.

Proof:

Let

$$R = \{ q \in G : q = (q_1, \ldots, q_n) \text{ with } q_i \in \mathbb{Q} \}$$

Then R is countable (why?). For each $q \in R$, we will find an integer n_q such that

$B_{\frac{1}{n_q}}(q) \subseteq G$ and

$$G = \bigcup_{q \in R} B_{\frac{1}{n_q}}(q).$$

The difficulty is choosing the correct radius $1/n_q$. What we want to do is let $B_{1/n_q}(q)$ be as large a ball as possible that still fits into G, which means we want to take n_q as small as we can. To make this precise, given $q \in R$ think about

$$S_q = \{ n \in \mathbb{N} : B_{\frac{1}{n}}(q) \subseteq G \}.$$

Since $q \in G$ & G is open, we know there's some ball $B_r(x) \subseteq G$, & therefore there are infinitely many $n > 0$ s.t. $B_{1/n} \subseteq G$. Since the n are positive integers, there's a smallest element of S_q, & this is the one we want: define

$$n_q = \min(S_q)$$

i.e., n_q is the smallest n s.t. $B_{1/n}(q) \subseteq G$.

Now we have to show that $G = \bigcup_{q \in R} B_{1/n_q}(q)$.

Since for each q we have $B_{1/n_q}(q) \subseteq G$, we certainly have

$$\bigcup_{q \in R} B_{1/n_q}(q) \subseteq G.$$

It's the opposite inclusion that is the hard part.

So, choose any $x \in G$. We have to show that there's a $q \in R$ s.t. $x \in B_{1/n_q}(q)$ To start, since $x \in G$ & G is open, there does some $n > 0$ s.t.

$$B_{\frac{1}{n}}(x) \subseteq G$$

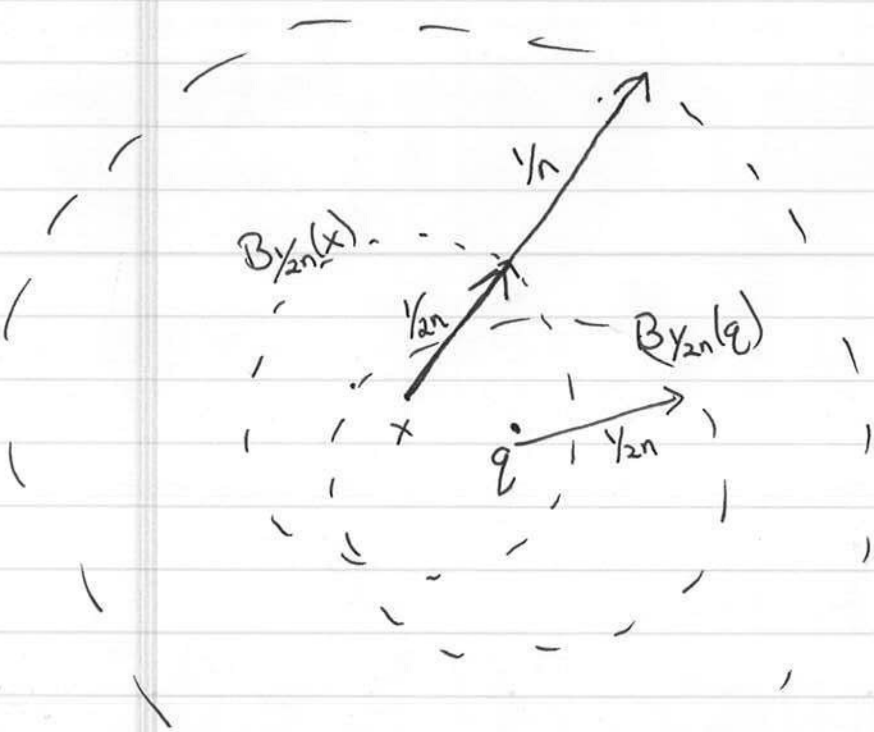
Note that this ball of radius $1/n$ is centered at x. We need to find a $q \in R$ so that x lies in the ball $B_{1/n_q}(q)$ centered at q.

Now by choosing rational $q_1, \ldots, q_p$ we can find a $q = (q_1, \ldots, q_p)$ as close as we like to $x = (x_1, \ldots, x_p)$. In particular, choose q close enough that we have

$$\|x - q\| < \frac{1}{2n}.$$

Then not only is $q \in B_{\frac{1}{2n}}(x)$ but also $x \in B_{\frac{1}{2n}}(q)$.

Claim: $B_{\frac{1}{2n}}(q) \subseteq G$



In fact, we claim that

$$B_{\frac{1}{2n}}(q) \subseteq B_{\frac{1}{n}}(x) \subseteq G$$

and we already know this inclusion.

To see this, choose any $y \in B_{\frac{1}{2n}}(q)$.

Then $\|y-q\| < \frac{1}{2n}$, so

$$\begin{aligned} \|y-x\| &= \|(y-q)+(q-x)\| \\ &\leq \|y-q\| + \|q-x\| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}, \end{aligned}$$

so indeed, $y \in B_{\frac{1}{n}}(x)$. Hence we have that $x \in B_{\frac{1}{2n}}(q) \subseteq G$. But by the definition of n_q, we have

$$B_{\frac{1}{2n}}(q) \subseteq B_{\frac{1}{n_q}}(q)$$

or in other words, $n_q \leq 2n$ (why?). Hence $x \in B_{\frac{1}{n_q}}(q)$, which is what we wanted to prove, i.e., we conclude that $G \subseteq \bigcup_{q \in R} B_{\frac{1}{n_q}}(q)$. $\blacksquare$