

TOPOLOGY OF CARTESIAN SPACES

8. Vector & Cartesian Spaces

Definition

A vector space is a set V together with two

operations:

$$+ : V \times V \longrightarrow V \quad \text{vector addition}$$
$$\cdot : \mathbb{R} \times V \longrightarrow V \quad \text{scalar mult.}$$

such that

$$(a) \quad x + y = y + x \quad \forall x, y \in V$$
$$(b) \quad (x + y) + z = x + (y + z) \quad \forall x, y, z \in V$$
$$(c) \quad \exists 0 \in V \quad \forall x \in V \quad x + 0 = x$$
$$(d) \quad \forall x \in V \quad \exists (-x) \in V \quad \text{st.} \quad x + (-x) = 0$$

$$(e) \quad 1x = x \quad \forall x \in V$$
$$(f) \quad a(bx) = (ab)x \quad \forall a, b \in \mathbb{R} \quad \forall x \in V$$

$$(g) \quad (a+b)x = ax + bx, \quad a(x+y) = ax + ay \quad \forall a, b \in \mathbb{R}, \quad \forall x, y \in V.$$

[\mathbb{R} can be replaced by any FIELD]

Examples: \mathbb{R}

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) : x_i \in \mathbb{R} \}$$

\mathbb{R}^S

If S is any set, then we define \mathbb{R}^S to be the set of all functions that map S into \mathbb{R} :

$$\mathbb{R}^S = \{f : f: S \rightarrow \mathbb{R}\}$$

This is a vector space. The rule for vector addition

is that if $f, g: S \rightarrow \mathbb{R}$ then $f+g: S \rightarrow \mathbb{R}$ is

given by

$$(f+g)(x) = f(x) + g(x), \quad x \in S$$

and if $f: S \rightarrow \mathbb{R}$ & $c \in \mathbb{R}$ then $cf: S \rightarrow \mathbb{R}$ is

$$(cf)(x) = cf(x), \quad x \in S.$$

We'll give several examples of such spaces.

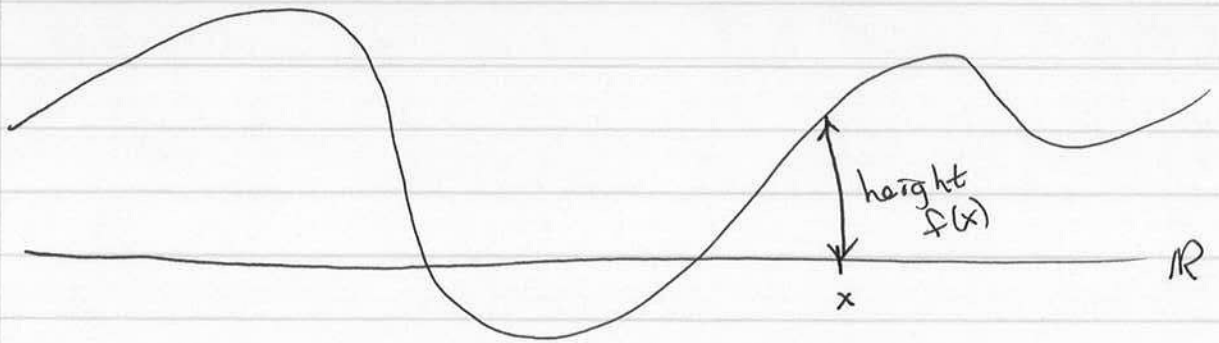
$\mathbb{R}^{\mathbb{R}}$

By definition, $\mathbb{R}^{\mathbb{R}}$ is the set of all functions that map real nos. to real nos.:

$$\mathbb{R}^{\mathbb{R}} = \{f : f: \mathbb{R} \rightarrow \mathbb{R}\}$$

We've seen this set before, we called it F or \mathcal{F} .

If $f \in \mathbb{R}^{\mathbb{R}}$ then we can describe it by its graph:



domain in \mathbb{R} , range is \mathbb{R}

Abusing notation, we have $e^x \in \mathbb{R}^{\mathbb{R}}$, $\sin x \in \mathbb{R}^{\mathbb{R}}$, etc.

Exercise

Show that $\mathbb{R}^{\mathbb{R}}$ is a vector space. The

operations are defined as follows: if $f, g \in \mathbb{R}^{\mathbb{R}}$

then $f+g$ is the function whose rule is

$$(f+g)(x) = f(x) + g(x), \quad x \in \mathbb{R}$$

Likewise, if $f \in \mathbb{R}^{\mathbb{R}}$ & $c \in \mathbb{R}$ then cf is the

function whose rule is

$$(cf)(x) = c f(x), \quad x \in \mathbb{R}.$$

The zero vector in $\mathbb{R}^{\mathbb{R}}$ is the function 0 (note

that we are abusing notation & writing the symbol 0

to denote a function) whose rule is

$$0(x) = 0, \quad x \in \mathbb{R}$$

↑
↑

Zero function
Zero number

(5)

$\mathbb{R}^{\mathbb{N}}$

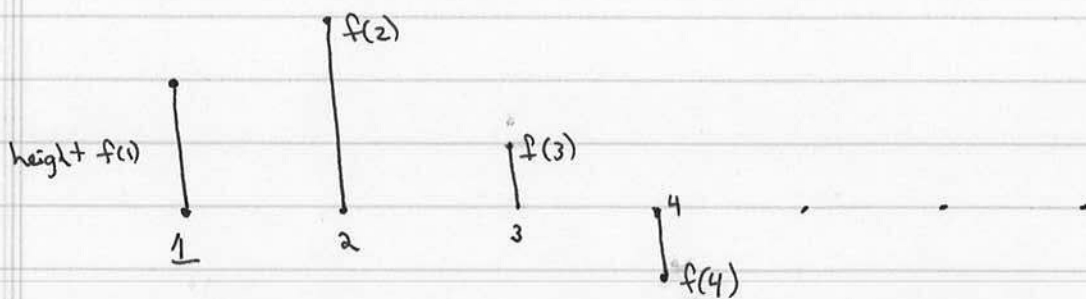
By definition, $\mathbb{R}^{\mathbb{N}}$ is the set of all functions that map \mathbb{N} into \mathbb{R} :

$$\mathbb{R}^{\mathbb{N}} = \{f : f: \mathbb{N} \rightarrow \mathbb{R}\}$$

Let f be an element of $\mathbb{R}^{\mathbb{N}}$, i.e., $f: \mathbb{N} \rightarrow \mathbb{R}$.

So f maps natural numbers to real numbers.

If we know what $f(1), f(2), f(3), \dots$ are, then we know f completely. We can think of f as a "discrete" or "digital" function:



domain is $\{1, 2, 3, \dots\}$ range is \mathbb{R}

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Space of sequences

Let

$$S = \left\{ x = (x_1, x_2, x_3, \dots) : x_1, x_2, \dots \in \mathbb{R} \right\}$$

That is, S is the set of all infinite sequences of real numbers.

Exercise

S is a vector space under the operations

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$$

$$c(x_1, x_2, \dots) = (cx_1, cx_2, \dots)$$

The zero vector is

$$0 = (0, 0, 0, \dots)$$

Note that we usually abuse notation & use the same symbol 0 to denote two different things:

it can be either the zero number or the zero sequence

We tell the difference by looking at the context in which

the symbol is used.

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$\mathbb{R}^{\mathbb{N}}$ and S are isomorphic

$\mathbb{R}^{\mathbb{N}}$ and S are really the same space in two different disguises.

$f \in \mathbb{R}^{\mathbb{N}} \Rightarrow f$ determines an associated sequence $(f(1), f(2), \dots)$

and

$x = (x_1, x_2, \dots) \in S \Rightarrow x$ determines an associated function $f: \mathbb{N} \rightarrow \mathbb{R}$ by

$$f(n) = x_n$$

Exercise

Make this precise as follows. Define

$$\begin{aligned} \varphi: \mathbb{R}^{\mathbb{N}} &\longrightarrow S \\ f &\longmapsto (f(1), f(2), \dots) \end{aligned}$$

That is, if $f: \mathbb{N} \rightarrow \mathbb{R}$ then $\varphi(f)$ is the sequence

$$\varphi(f) = (f(1), f(2), \dots)$$

Prove the following.

a. φ is a bijection

b. φ is linear, i.e., it preserves the vector space operations: $\forall f, g \in \mathbb{R}^N$ and $\forall c \in \mathbb{R}$,

$$\varphi(f+g) = \varphi(f) + \varphi(g)$$

$$\varphi(cf) = c\varphi(f)$$

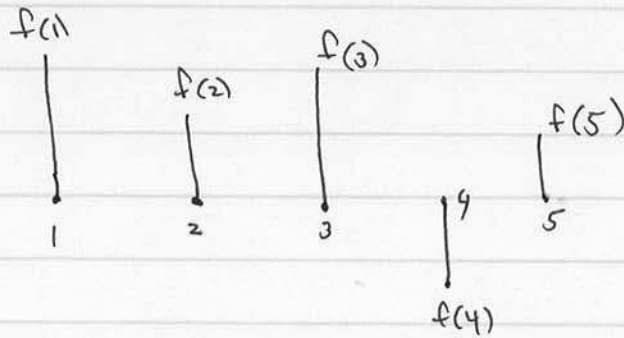
A linear bijection between vector spaces is called an isomorphism. Thus $\varphi: \mathbb{R}^N \rightarrow S$ is an isomorphism, and we say that \mathbb{R}^N and S are isomorphic.

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$$\mathbb{R}^{\{1,2,3,4,5\}}$$

By definition,

$$\mathbb{R}^{\{1,2,3,4,5\}} = \left\{ f : f : \{1,2,3,4,5\} \rightarrow \mathbb{R} \right\}$$



In ~~contrast~~ contrast, \mathbb{R}^5 is a space of ordered 5-tuples:

$$\mathbb{R}^5 = \left\{ x = (x_1, x_2, x_3, x_4, x_5) : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}$$

Exercise

Show $\mathbb{R}^{\{1,2,3,4,5\}}$ & \mathbb{R}^5 are isomorphic. What is

the isomorphism?

Thus, a sequence $(x_1, x_2, x_3, x_4, x_5)$ "really" is a function with domain $\{1, 2, 3, 4, 5\}$ and function

values

$$f(1) = x_1$$

$$f(2) = x_2$$

$$f(3) = x_3$$

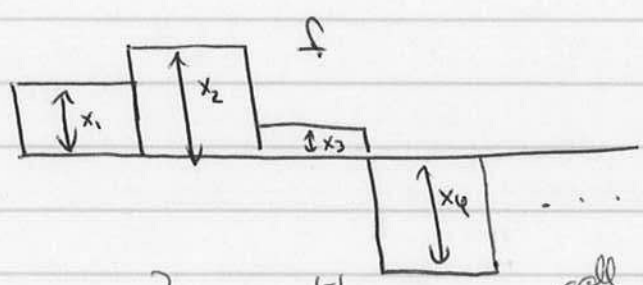
$$f(4) = x_4$$

$$f(5) = x_5.$$

Sequences can be viewed as discrete functions

(x_1, x_2, \dots, x_n) represents $f: \{1, \dots, n\} \rightarrow \mathbb{R}$

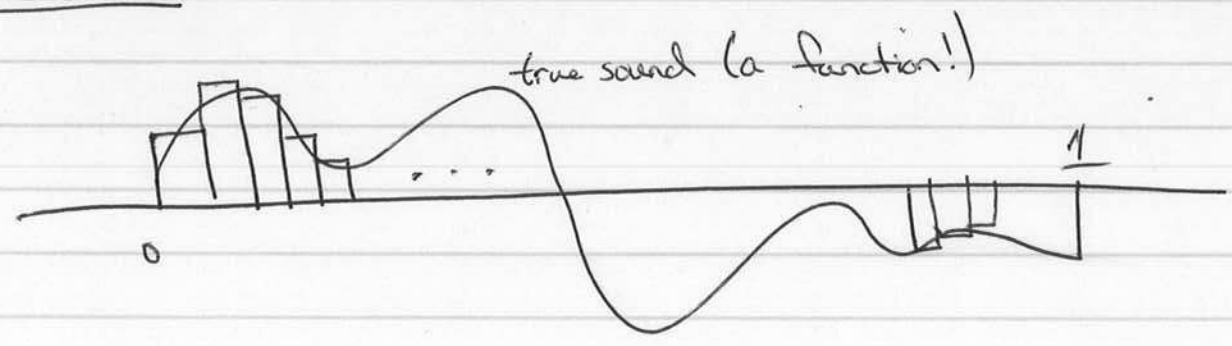
$$\begin{aligned}
 f(1) &= x_1 \\
 f(2) &= x_2 \\
 &\vdots \\
 f(n) &= x_n
 \end{aligned}$$



$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\} \xleftrightarrow[\text{correspondence}]{1-1} S = \{\text{all } f: \{1, \dots, n\} \rightarrow \mathbb{R}\}$$

More than just 1-1 corr: Addition, mult in \mathbb{R}^n reflected in add., mult in S .

Ex. CD sound.



1 sec CD sound stored as sequence of 44,000 nos

= one vector in $\mathbb{R}^{44,000}$

= one function in $\mathbb{R}^{\{1, \dots, 44,000\}}$

The true 1 sec of CD sound can be viewed as a function f on $[0, 1]$

$$f \in \mathbb{R}^{[0, 1]}$$

Norms

If V is a vector space then a function $\|\cdot\|: V \rightarrow \mathbb{R}$

is a norm on V if

a. $\|x\| \geq 0 \quad \forall x \in V$

b. $\|x\| = 0 \iff x = 0$

c. $\|cx\| = |c| \|x\| \quad \forall c \in \mathbb{R}, x \in V$

d. Triangle Inequality: $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

Example

The norm that will mostly concern us is the Euclidean norm on \mathbb{R}^p , which is defined by

$$\|x\| = (x_1^2 + \dots + x_p^2)^{1/2}, \quad x \in \mathbb{R}^p.$$

However, there are infinitely many other norms on \mathbb{R}^p

If q is any real number in the range $1 \leq q < \infty$, then

$$\|x\|_q = (|x_1|^q + \dots + |x_p|^q)^{1/q}, \quad x \in \mathbb{R}^p$$

is a norm on \mathbb{R}^p . Another norm is

$$\|x\|_\infty = \max \{ |x_1|, \dots, |x_p| \}.$$

Exercise: Prove that $\|\cdot\|_q$ is a norm when

$q=1$ or $q=\infty$ (these are the easy ones, the proof for other values of q is trickier).

Example

Let $S = \{x = (x_1, x_2, \dots) : x_k \in \mathbb{R}\}$ be

the space of all infinite sequences of real numbers. Is

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|, \quad x \in S$$

a norm on S ?

The answer is no, because $\|x\|_1$ need not be finite, and by definition a norm on S must take only (nonnegative) real values.

One way to "fix" \mathcal{L}_2 is to consider a subspace of S : define

$$l^1 = \left\{ x = (x_1, x_2, \dots) \in S : \sum_{k=1}^{\infty} |x_k| < \infty \right\}$$

Examples: $(1, 1, 1, \dots) \notin l^1$

$(1, \frac{1}{2}, \frac{1}{3}, \dots) \notin l^1$

$(1, \frac{1}{4}, \frac{1}{9}, \dots) \in l^1$

Exercise l^1 is a vector space, and

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k|, \quad x \in l^1$$

is a norm on l^1 .

Likewise, given any $1 \leq q < \infty$, define

$$l^q = \left\{ x = (x_1, x_2, \dots) \in S : \sum_{k=1}^{\infty} |x_k|^q < \infty \right\}$$

It is not so easy to prove when $q \neq 1$, but it is true that l^q is a vector space, and

$$\|x\|_q = \left(\sum_{k=1}^{\infty} |x_k|^q \right)^{1/q}, \quad x \in l^q$$

is a norm on l^q .

Exercise: Show that

$$l^\infty = \left\{ x = (x_1, x_2, \dots) \in S : \sup_k |x_k| < \infty \right\}$$

is a vector space, and

$$\|x\|_\infty = \sup_k |x_k|, \quad x \in S$$

is a norm on l^∞ .

Show that if $1 < q < \infty$, then

$$l^1 \subsetneq l^q \subsetneq l^\infty.$$

Inner Products and Their Norms

Recall the dot product of vectors in \mathbb{R}^p :

$$X \cdot y = x_1 y_1 + \dots + x_p y_p.$$

This is an example of an inner product.

Definition

An inner product on a vector space V is a

function

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

that satisfies

a. $x \cdot x \geq 0$

b. $x \cdot x = 0 \iff x = 0$

c. $x \cdot y = y \cdot x$

d. $x \cdot (y + z) = x \cdot y + x \cdot z$

e. $(cx) \cdot y = c(x \cdot y)$.

Other notations

On spaces over \mathbb{R} or \mathbb{R}^p , we usually write $\langle x, y \rangle$ or $[x, y]$ instead of $x \cdot y$ for an inner product. Even on \mathbb{R}^p we may use this notation if we are using an inner product different than the standard dot product.

Exercise

Let w_1, \dots, w_p be fixed positive real numbers. Show that

$$\langle x, y \rangle = x_1 y_1 w_1 + \dots + x_p y_p w_p$$

is an inner product on \mathbb{R}^p .

Example

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k, \quad x, y \in \ell^2$$

is an inner product on ℓ^2 .

Claim: All inner products lead to norms.
Not all norms lead to inner products.

Theorem

If V is an inner product space then $\|x\| = (x \cdot x)^{1/2}$ is a norm on V . Further,

$$x \cdot y \leq \|x\| \|y\| \quad (\text{Weak Schwarz Inequality})$$

and $\forall x, y \neq 0$,

~~and~~ $\|x \cdot y\| = \|x\| \|y\|$ if & only if $x = cy$ for some $c > 0$.

Proof:

(i) $\|x\| = (x \cdot x)^{1/2} \geq 0 \quad \checkmark$

(ii) $\|x\| = (x \cdot x)^{1/2} = 0 \Leftrightarrow x \cdot x = 0 \Leftrightarrow x = 0 \quad \checkmark$

(iii) $\|ax\|^2 = (ax \cdot ax) = a(x \cdot ax) = a^2(x \cdot x)$

So $\|ax\| = |a| \|x\|$.

(iv) Set $z = ax - by$. Then

$$0 \leq z \cdot z = (ax - by) \cdot (ax - by)$$

$$= a^2 x \cdot x - 2ab x \cdot y + b^2 y \cdot y$$

$$= a^2 \|x\|^2 - 2ab x \cdot y + b^2 \|y\|^2$$

For the specific case of $a = \|y\|$ & $b = \|x\|$,

Q3 says that

$$0 \leq \|y\|^2 \|x\|^2 - 2 \|y\| \|x\| x \cdot y + \|x\|^2 \|y\|^2$$

~~...~~

$$= 2 \|x\| \|y\| (\|x\| \|y\| - x \cdot y).$$

Since $\|x\|, \|y\| \geq 0$, we have $\|x\| \|y\| - x \cdot y \geq 0$.

~~...~~

Hence $x \cdot y \leq \|x\| \|y\|$ is true. Therefore

$$\begin{aligned} \|x+y\|^2 &= (x+y) \cdot (x+y) \\ &= x \cdot x + 2x \cdot y + y \cdot y \\ &\leq \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2, \end{aligned}$$

so $\|x+y\| \leq \|x\| + \|y\|$.

Exercise: Show $x \cdot y = \|x\| \|y\| \iff x = cy$ with $c > 0$.

~~XXXXXXXXXX~~

Thus every inner product has an associated norm.

Corollary

$$|x \cdot y| \leq \|x\| \|y\| \quad (\text{Schwarz Inequality})$$

And, $\forall y \neq 0, |x \cdot y| = \|x\| \|y\| \iff x = cy.$

Not all norms lead to inner products!

Ex: There is no inner product on \mathbb{R}^p st.

$$\|x\|_1 = (x \cdot x)^{1/2}$$

But, \mathbb{R}^p does have some norm that comes from an inner product (\mathbb{R}^p is a Hilbert space)

Exercise
8.11

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

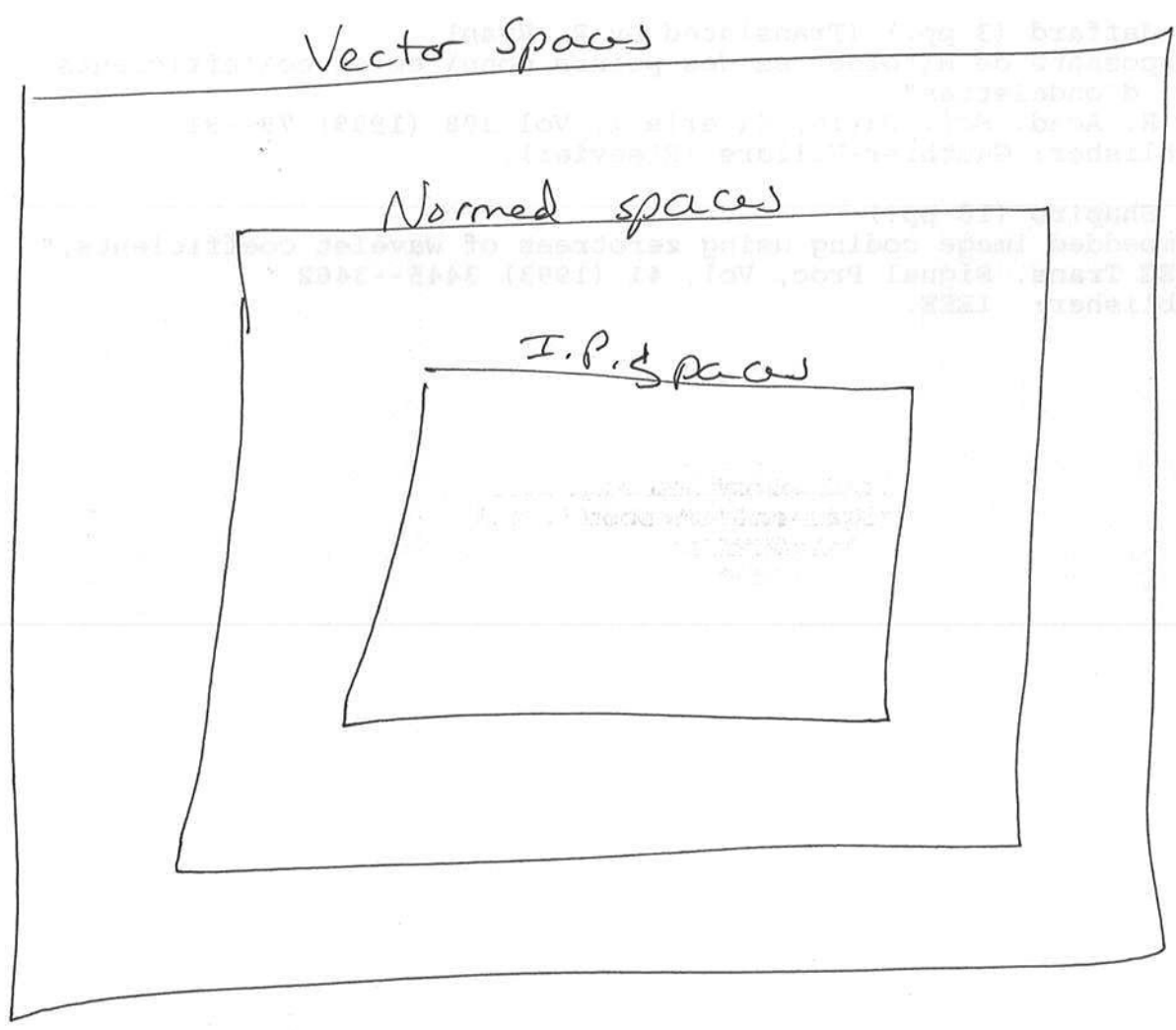
Ex: There is no norm on \mathcal{L}^1 that comes from an inner product.

Fails parallelogram Law

Contrast $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$ on \mathcal{L}^2

does come from an inner product.

V i.p. space $\implies \|x\| = (x \cdot x)^{1/2}$ is a norm
 \forall i.p. $x \cdot y$



Some spaces can have many norms or i.p.s.

We'll mostly use the
Euclidean norm on \mathbb{R}^p

Note $\|x\| = \|x\|_2 = \sqrt{x_1^2 + \dots + x_p^2}$
 Unless otherwise specified, this is the norm we mean. \leftarrow

Open ball:

$$B_r(x) = \{y \in \mathbb{R}^p : \|x - y\| < r\}$$

Notation Alert!

\leftarrow NOT
 Ball notation !!

Closed ball

$$\overline{B}_r(x) = \{y \in \mathbb{R}^p : \|x - y\| \leq r\}$$

Sphere:

$$\overline{B}_r(x) \setminus B_r(x) = \{y \in \mathbb{R}^p : \|x - y\| = r\}$$

Note: $p=1$ balls are intervals

$$B_r(x) = (x-r, x+r)$$