7. Cuts, Intervals, & the Cantor Set

Purpose of Cuts

Used to construct the real nos. from the set theory axioms - we'll take the construction of $\mathbb{R}$ as given

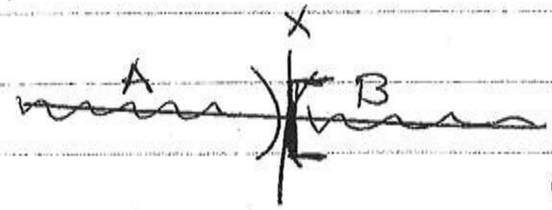
Definition

Let $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$. Then (A, B) is a cut if:

$$A \cap B = \emptyset, \quad A \cup B = \mathbb{R}, \quad a \in A \;\&\; b \in B \Rightarrow a < b.$$

Ex: $A = (-\infty, x) = \{y : y < x\}$

$B = [x, \infty) = \{z : z \geq x\}$



Claim: Each cut is determined by a unique real number.

Cut Property

If (A, B) is a cut in $\mathbb{R}$ then $\exists$ unique $\xi \in \mathbb{R}$ st.

$a \leq \xi \;\; \forall a \in A \quad \& \quad b \geq \xi \;\; \forall b \in B$

Proof.

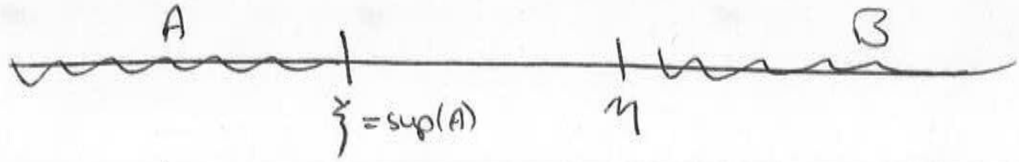
A is bounded above (by any element of B)

B " " below " " A

Set $\xi = \sup A$. Then $a \leq \xi \;\; \forall a \in A$.

Since every element of B is an upper bound for A & ξ is the least upper bound for A, we must have $\xi \leq b$ for every $b \in B$

UNIQUE?



Suppose there also existed η st. $\eta \geq a \ \forall a \in A$
$\eta \leq b \ \forall b \in B$.

Then η is an upper bound for A so $\xi \leq \eta$

Suppose $\xi < \eta$. Then $\exists\, x$ st. $\xi < x < \eta$.

$$\text{Then} \quad \begin{array}{lll} x > a & \forall a \in A \Rightarrow & x \notin A \\ x < b & \forall b \in B \Rightarrow & x \notin B \end{array} \quad \text{Contradiction } A \cup B = \mathbb{R}.$$

Hence $\xi = \eta$ is the only possibility. $\blacksquare$

Cells, Rays, Intervals: Particular subsets of $\mathbb{R}$

Rays: (a,∞) open, $[a,\infty)$ closed, $(-\infty,a)$ open, $(-\infty,a]$ closed

Cells: (a,b) open, $[a,b]$ closed, $(a,b]$ $[a,b)$ half-open

Interval = cell, ray, or $\mathbb{R}$

Unit cell $[0,1]$

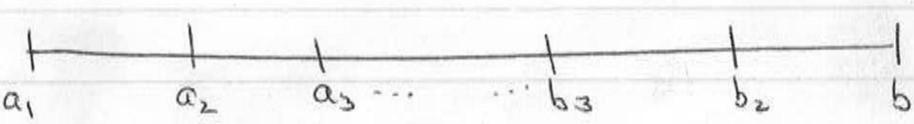
Intervals $I_1, I_2, \dots$ are nested if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

Ex: $I_n = (0, \frac{1}{n})$ is nested. Note $\bigcap I_n = \emptyset$
$J_n = [0, \frac{1}{n}]$ " " $\bigcap J_n = \{0\}$
$K_n = [-\frac{1}{n}, 1+\frac{1}{n}]$ " " $\bigcap K_n = [0,1]$.

Nested Cells Property

Let $I_n = [a_n, b_n]$ be such that the seq. $I_1, I_2, \dots$ is nested.

Then $\bigcap I_n \neq \emptyset$.



Proof:

Since $[a_n, b_n] = I_n \subseteq I_1 = [a_1, b_1]$ we must have

$a_n \geq a_1 \ \forall n$. Let $\xi = \sup\{a_n : n \in \mathbb{N}\}$.

Claim: $\xi \leq b_n \ \forall n$.

Suppose not. Then $\exists m$ st. $b_m < \xi$.

But $\xi = \sup\{a_n\}$ so $\exists n$ st. $b_m < a_n \leq \xi$.

If $m \geq n$: $b_m < a_n \leq a_m$ Contradiction

If $m \leq n$: $b_n \leq b_m < a_n$ Contradiction

So $\xi \leq b_n \ \forall n$. Hence $\xi \in [a_n, b_n] \ \forall n$,

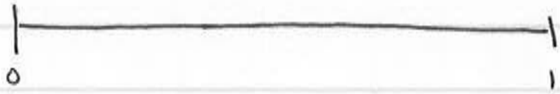
so $\xi \in \bigcap [a_n, b_n]$. ∎

Note: Nested Cells Property can fail if the sets are not closed!

$$\bigcap_{n=1}^{\infty} \left(0, \tfrac{1}{n}\right) = \emptyset$$

Cantor Set

Define sets $F_0, F_1, F_2, \ldots$ by the following procedure.



$F_0 = [0,1]$

Now create F_1 by removing the "middle third" from F_0:

$F_1 = [0, 1/3] \cup [2/3, 1]$

Note F_1 consists of 2 intervals of length $\frac{1}{3}$ each.

Now create F_2 by removing the middle third from each of the intervals that comprise F_1:

$F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

Note F_2 consists of $4 = 2^2$ intervals each of length $\frac{1}{9} = \frac{1}{3^2}$.

Keep repeating this process

F_3

F_3 consists of $8 = 2^3$ intervals each of length $\frac{1}{27} = \frac{1}{3^3}$.

Keep going ... F_n looks like

$$F_n = \bigcup_{\substack{2^n \text{ particular} \\ \text{values of } k}} \left[\frac{k}{3^n}, \frac{k+1}{3^n} \right]$$

i.e., F_n is a union of 2^n intervals, each of length $\frac{1}{3^n}$, and if we wanted, we could specifically identify the k's in the union above.

Note the sets F_n are <u>nested</u>, i.e.,

$$F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \cdots$$

Also,

total length of intervals in F_0 $= 1 \times 1 = 1$

" " " " " F_1 $= 2 \cdot \frac{1}{3} = \frac{2}{3}$

" " " " " F_2 $= 4 \cdot \frac{1}{9} = \frac{4}{9} = \left(\frac{2}{3}\right)^2$

$\vdots$

" " " " " F_n $= 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$

$\vdots$

What if we do this "forever" – is there anything left?

What is left?

Stuff in F_0 but not F_1 is removed at the first step – only things in both F_0 & F_1 remain.

Stuff in F_1 but not F_2 is removed at the 2nd step – only things in all of F_0, F_1, F_2 remain

⋮

Only stuff in $F_1 \cap F_2 \cap \cdots \cap F_n$ remains at the nth step

⋮

Stuff that remains after every step is

$$F = \bigcap_{n=1}^{\infty} F_n$$

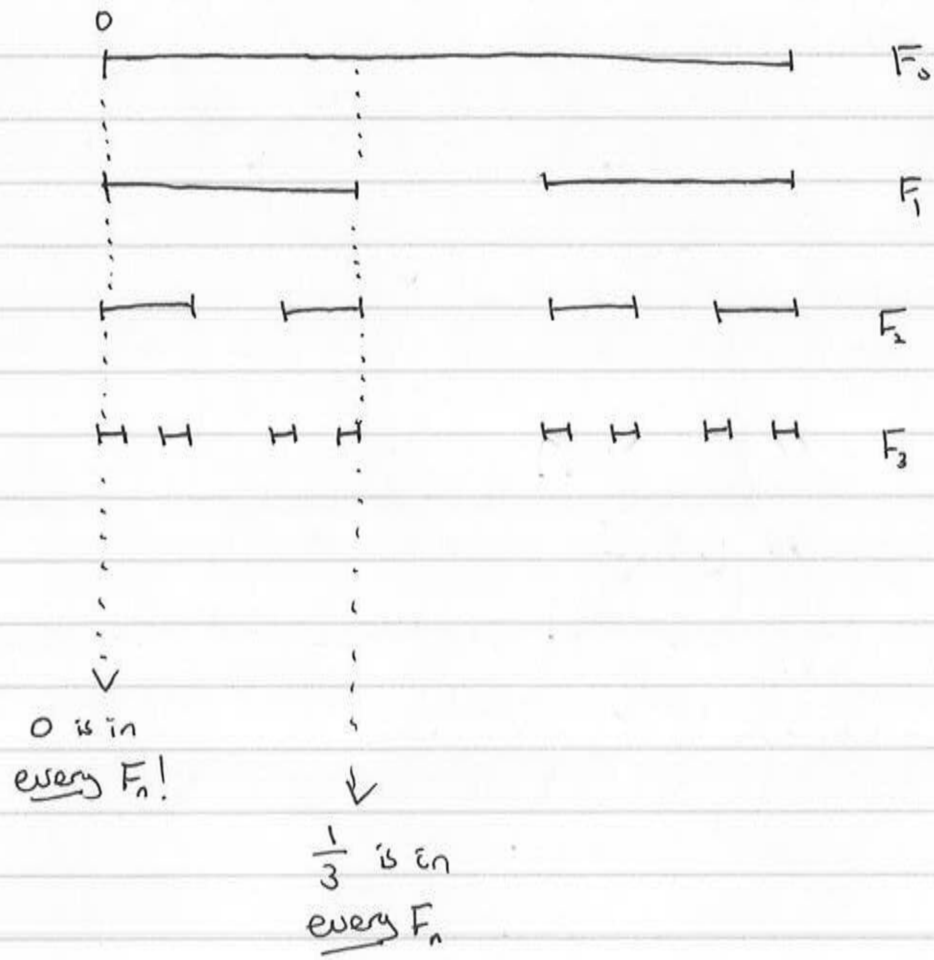
This is the Cantor set.

Q. Is there anything in the Cantor set ??

We have

$$F \subseteq F_n \quad \text{for every } n,$$

but perhaps F is just the empty set.??



Each F_n consists of 2^n intervals. There are

$2 \cdot 2^n = 2^{n+1}$ endpoints of these intervals.

For each n, the 2^{n+1} endpoints of the intervals that comprise F_n belong to F!

So, F does contain infinitely many points.

There are countably many endpoints of the intervals in the F_n, so F has at least countably many points in it.

Q. Are these <u>all</u> the elements of F?

To answer this, we will use <u>ternary expansions</u>.

<u>Note</u>

F itself does NOT have any "endpoints." $\frac{1}{3}$ is not an "endpoint" of F. $\frac{1}{3}$ <u>is</u> an endpoint of an interval in F_1, but F itself contains <u>no intervals</u>

Ex: Prove that $\nexists$ interval (a,b) such that $(a,b) \subseteq F$.

Decimal Expansions

Recall that when we write

$$x = 0.d_1 d_2 d_3 \cdots \qquad \text{(each } d_k \text{ is one of } 0,1,\dots,9)$$

We mean

$$x = \sum_{k=1}^{\infty} \frac{d_k}{10^k}.$$

Exercise: This series converges for every choice of digits d_k. Why?

Example

Let's prove that $\frac{1}{3} = .333\cdots$. That is, let

$$x = \sum_{k=1}^{\infty} \frac{3}{10^k}$$

and let's prove that $x = \frac{1}{3}$. We have

$$\begin{aligned} 10x &= 10 \sum_{k=1}^{\infty} \frac{3}{10^k} & &= 10 \cdot \left(\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots \right) \\ &= \sum_{k=1}^{\infty} \frac{3}{10^{k-1}} & &= 3 + \frac{3}{10} + \frac{3}{100} + \cdots \\ &= \frac{3}{10^0} + \sum_{k=1}^{\infty} \frac{3}{10^k} & &= 3 + \left(\frac{3}{10} + \frac{3}{100} + \cdots \right) \\ &= 3 + x & &= 3 + x \end{aligned}$$

So, whatever x is, it satisfies the equation

$$10x = 3 + x$$

Hence $\quad 9x = 3 \quad$ so $\quad x = \frac{1}{3}.$

Exercise

Some numbers have two decimal expansions. Show that

$$1.000\ldots = 0.999\cdots$$

That is, show that

$$1 = \sum_{k=1}^{\infty} \frac{9}{10^k}$$

(use the same technique as in the example).

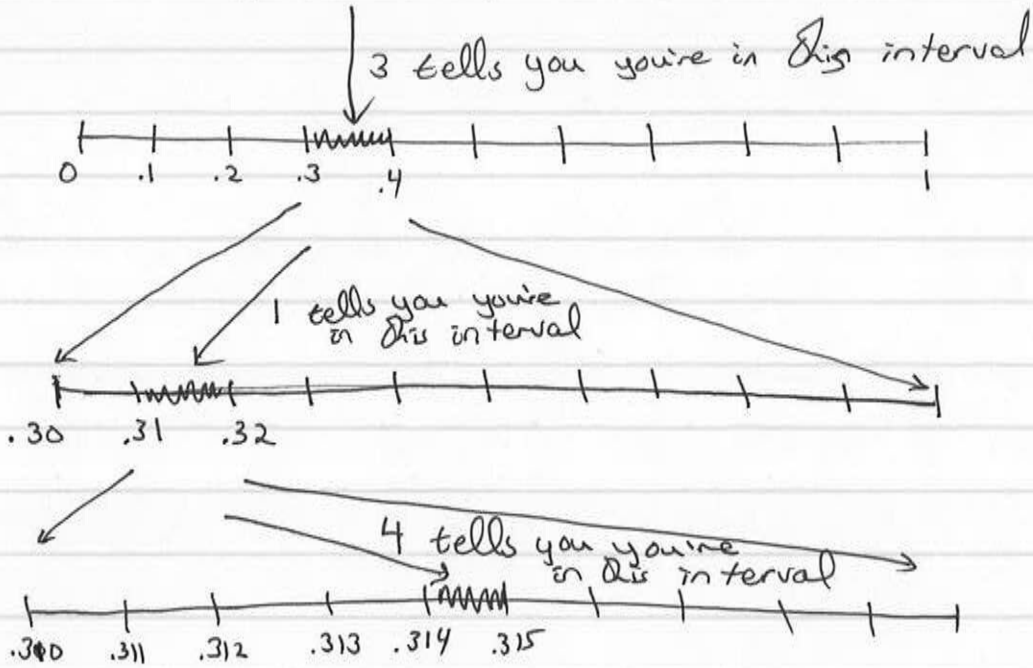
Remark:

The numbers that have two expansions are the ones that have an expansion ending in ∞ many zeros. E.g.,

$$.7346 = .7345999\cdots$$

What does the decimal expansion tell you?

Consider $\quad X = .314159\ldots$



etc. At each stage you divide the interval into 10 parts & identify which part you are in. For the "endpoint" numbers, like .3, .32, .314, ... there are two choices of subinterval to go in, & this gives the two decimal expansions.

Isn't this reminiscent of the Cantor set construction, except using tenths instead of thirds?

Ternary Expansions

There's nothing special about the number 10, we can base "decimal expansions" on any positive integer. In particular, ternary expansions use base 3.

We write

$$x = 0.d_1 d_2 d_3 \dots \text{ base } 3 \qquad (\text{each } d_k \text{ is one of } 0, 1, 2)$$

to mean

$$x = \sum_{k=1}^{\infty} \frac{d_k}{3^k}.$$

Exercise: $\frac{1}{3} = .1000\dots \text{ base } 3 = .0222\dots \text{ base } 3.$

Some numbers have two ternary expansions; in this case one expansion ends with infinitely many zeros, and the other with infinitely many 2's.

Exercise

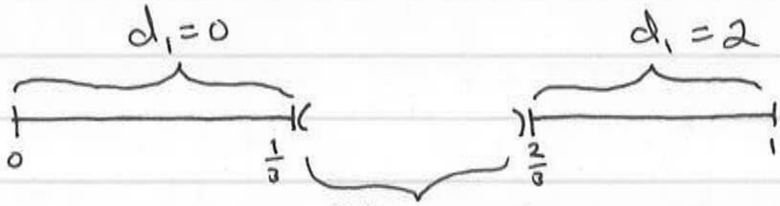
$$\frac{1}{4} = .020202\ldots \quad \text{base } 3$$

What points are in the Cantor Set?

Write

$$X = 0.d_1 d_2 d_3 \cdots \quad \text{base } 3.$$

Stage 1:



These points are removed. They all have $d_1 = 1$.

The endpoints, $\frac{1}{3}$ & $\frac{2}{3}$ have two ternary expansions:

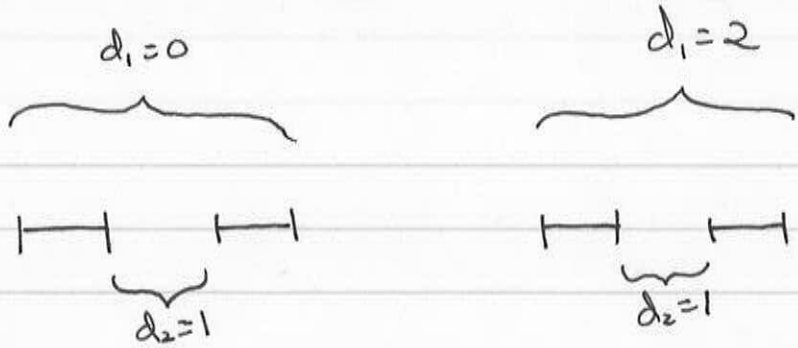
$$\frac{1}{3} = 0.1000\ldots = 0.0222\cdots$$

$$\frac{2}{3} = 0.2000\cdots = 0.1222\cdots$$

They are not removed.

So: As long as x has a ternary expansion that has either $d_1 = 0$ or $d_1 = 2$, then x is not removed in Stage 1.

Stage 2:



In stage 2, all remaining points that have $d_2=1$ (except for the endpts). Precisely, as long as there's a ternary expansion that has either $d_2=0$ or $d_2=2$, the point is not removed.

Now repeat. If

$$x = 0.d_1d_2d_3\cdots \text{ base } 3$$

& $d_1, \dots, d_n = 0$ or 2 (for at least one of x's ternary expansions) then x_n is not removed in the first n stages, ie, $x \in F_n$.

Conclude: if there's a ternary expansion such that <u>every</u> d_k is 0 or 2, then x is in every F_n and hence is in $F = \bigcap F_n$.

Example: $\frac{1}{4} = 0.020202\cdots$ base 3

so $\frac{1}{4} \in F$, even though $\frac{1}{4}$ is not an ~~[illegible]~~ endpoint of any F_n!

How many points are in F?

Each $x \in F$ can be written

$$x = 0.d_1 d_2 d_3 \cdots \text{ base } 3$$

where each d_k is 0 or 2. Define

$$d_k' = \begin{cases} 1 & \text{if } d_k = 2 \\ 0 & \text{if } d_k = 0 \end{cases}$$

Define

$$f(x) = 0.d_1' d_2' d_3' \cdots \text{ base } 2 = \sum_{k=1}^{\infty} \frac{d_k'}{2^k}.$$

<u>Example</u>

$$f\left(\tfrac{1}{4}\right) = f(0.020202\cdots \text{ base } 3)$$

$$= 0.010101\cdots \text{ base } 2$$

$$= \frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \cdots.$$

$$= \sum_{k=1}^{\infty} \frac{1}{4^k}$$

$$= \frac{1}{3}.$$

<u>Claim:</u> f ~~[illegible]~~ maps F <u>onto</u> $[0,1]$.

<u>Proof:</u>

Suppose $y \in [0,1]$. Then y has a base 2 expansion

$$y = 0.e_1 e_2 e_3 \cdots \text{ base } 2 \qquad e_k = 0 \text{ or } 1.$$

Set

$$d_k = \begin{cases} 2 & \text{if } e_k = 1 \\ 0 & \text{if } e_k = 0 \end{cases}$$

and

$$x = .d_1 d_2 d_3 \cdots \text{ base } 3.$$

Then x has a ternary expansion that contains only 0's & 2's, so $x \in F$. Further, by definition of f we have $f(x) = y$. Hence f maps onto all of $[0,1]$.

Exercise

Even without knowing whether f is injective, show that this implies that F is uncountable. More generally show that if X, Y are sets & Y is uncountable, if $\exists$ surjection $f: X \to Y$ then X must be uncountable.

THE CANTOR SET IS UNCOUNTABLE!

Compare: The endpoints of the F_n sets are in F, but there are only countably many of those. Hence, "most" points in F are not endpoints of any F_n!

How "long" is the Cantor Set?

Note: Each F_n is a union of finitely many intervals, but F, whatever it is, is not a union of intervals. Hence we can't really talk about F's "total length." Still, if it did make sense to talk about F's "length", what would it be?

$$F \subseteq F_0 \quad \text{so} \quad \text{length}(F) \leq \text{length}(F_0) = 1$$

$$F \subseteq F_1 \quad \text{so} \quad \text{length}(F) \leq \text{length}(F_1) = \frac{2}{3}$$

$$\vdots$$

$$F \subseteq F_n \quad \text{so} \quad \text{length}(F) \leq \text{length}(F_n) = \left(\frac{2}{3}\right)^n$$

$$\vdots$$

This is true for every n, so since $\left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$ we can only have

$$\text{length}(F) = 0 \quad !$$

Q. Is it possible for a nonempty set to have "length 0"?

[We still haven't defined exactly what "length" of an arbitrary set means — this is what measure theory is about, taught at GA Tech in the graduate course MATH 6327.]

A. Yes, there are lots of nonempty sets that have "length" zero.

Examples: a single point $\{x\}$

Finitely many points: $\{x_1, \ldots, x_n\}$

Countably many points: $\{x_1, x_2, \ldots\}$

Why should a set like $S = \{x_1, x_2, \ldots\}$ have length zero?

Intuitively, since $S = \bigcup_{n=1}^{\infty} \{x_n\}$, we should hope that

$$(*) \quad \text{length}(S) = \sum_{n=1}^{\infty} \text{length}(\{x_n\}) = \sum_{n=1}^{\infty} 0 = 0$$

Warning: What seems intuitively clear in measure theory is not always true. In particular, there is no way to extend the definition of length of an interval to length of arbitrary subsets of $\mathbb{R}$ so that

$$A \cap B = \emptyset \implies \text{length}(A \cup B) = \text{length}(A) + \text{length}(B) \,!$$

That is, there exist <u>disjoint</u> sets A, B such that the "length" of their union is <u>greater</u> than the sum of their "lengths."

Even so, equation (*) <u>is</u> valid and we conclude that

$$\text{if } S \subseteq \mathbb{R} \text{ is countable then } \text{length}(S) = 0.$$

This is not a surprise — the surprise is that $\exists$ uncountable sets with length zero!

THE CANTOR SET IS AN UNCOUNTABLE SET THAT HAS ZERO TOTAL LENGTH.

The Cantor set is a very interesting set & we will see more of its ~~[illegible]~~ surprising properties later.

Exercise

Since the set $\mathbb{Q}$ of all rationals is countable,

we can make a list of all the rational numbers in $\mathbb{R}$.

Let

$$\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$$

be one such ordering of $\mathbb{Q}$. Fix any $\varepsilon > 0$, and

let I_k be the interval of length $\varepsilon/2^k$ centered

at r_k:

$$I_k = \left[r_k - \frac{\varepsilon}{2^{k+1}}, r_k + \frac{\varepsilon}{2^{k+1}}\right].$$

Let $E = \bigcup_{k=1}^{\infty} I_k$. Show that

a. $\mathbb{Q} \subseteq E$

b. $\text{length}(E) \leq \varepsilon$ (note the I_k might overlap, so only can say $\leq$, not $=$).

Try to explain how this is possible - the rationals are

"dense" in $\mathbb{R}$ yet have been covered by a set with

total length at most ε. What does $\mathbb{R} \setminus E$ look like – does it contain any intervals?

<u>Exercise</u>

Make a Cantor-like set by removing at each stage a set of a different relative length. For example, instead of always removing the middle $\frac{1}{3}$ of the interval, you could remove the middle $\frac{1}{n}$ th at stage n. Show that by choosing the correct relative lengths you can create a Cantor-like set C that is uncountable, has <u>positive</u> total length, yet contains no intervals.