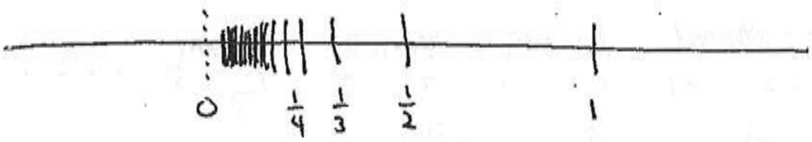
6. The Completeness Property of $\mathbb{R}$

Definition. Given $S \subseteq \mathbb{R}$.

(a) $u \in \mathbb{R}$ is an upper bound for S if $s \le u$ for all $s \in S$

(b) $\ell \in \mathbb{R}$ " " lower bound for S if $\ell \le s$ " " $s \in S$

Example $S = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$



Upper bounds: $1, 1.1, 2, \pi, 10, \ldots$

Lower bounds: $0, -0.001, -1, \ldots$

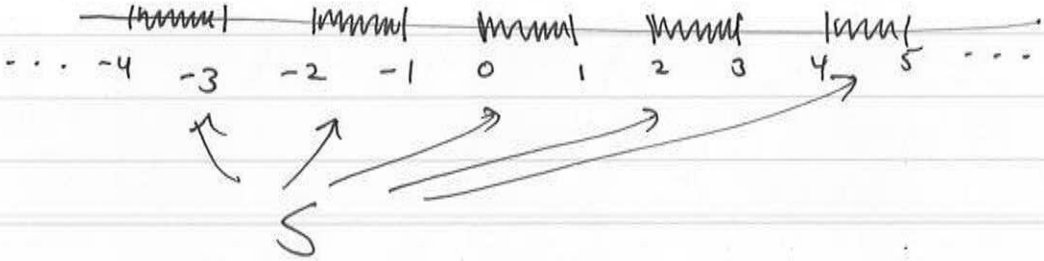
Doesn't matter whether the u.b. or l.b. is in S or not.

Example $S = (0, \infty)$ No upper bounds
Lower bounds: 0, anything < 0.

Definition Given $S \subseteq \mathbb{R}$.

S is bounded above if it has an upper bound.
" " bounded below " " " " lower bound.
" " bounded " " " both an upper & lower bound.
" " unbounded "

Example $S = \bigcup_{k=-\infty}^{\infty} [2k, 2k+1]$ is unbounded



<u>Definition</u> Given $S \subseteq \mathbb{R}$.

(a) If S is bounded above then $u \in \mathbb{R}$ is a <u>supremum</u> or <u>least upper bound</u> for S if

i. u is an upper bound for S

ii. if v is any other upper bound for S then $u \leq v$.

(b) <u>infimum</u> or <u>greatest lower bound</u> is analogous.

<u>Example</u> $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

$u = 1$ is the supremum. Write $\sup S = 1$

$l = 0$ is the infimum Write $\inf S = 0$.

Exercise: <u>Prove</u> this! [Exercises are not to hand in]

Examples of how to prove inequalities involving sup's or inf's:

Lemma

Assume that $f, g : \mathbb{R} \to \mathbb{R}$ have bounded ranges, i.e., $R(f)$ & $R(g)$ are bounded subsets of $\mathbb{R}$.

If $f(x) \le g(x)$ for every x, then

$$\inf\{f(x) : x \in \mathbb{R}\} \le \inf\{g(x) : x \in \mathbb{R}\}$$

& $$\sup\{f(x) : x \in \mathbb{R}\} \le \sup\{g(x) : x \in \mathbb{R}\}$$

Proof:

Let
$$u = \sup\{f(x) : x \in \mathbb{R}\},$$
$$v = \sup\{g(x) : x \in \mathbb{R}\}.$$

(Remark: Giving complicated things simple names can help a lot)

We want to prove that $u \le v$. Since v is an upper bound for the set $\{g(x) : x \in \mathbb{R}\}$, we have

$$f(x) \le g(x) \le v \quad \text{for all } x.$$

Hence v is an upper bound for the set $\{f(x) : x \in \mathbb{R}\}$.

But the least upper bound for this set is u,

so we conclude $u \le v$.

Exercise: Prove the inequality involving inf's. ∎

NOTE: The domain of f, g could just as well be any set X in this lemma.

Another example:

<u>Lemma</u>

Suppose $f : X \to \mathbb{R}$ has bounded range, and let $a \in \mathbb{R}$. Then

$$\sup\{f(x) + a : x \in X\} = \sup\{f(x) : x \in X\} + a$$
$$\inf\{f(x) + a : x \in X\} = \inf\{f(x) : x \in X\} + a.$$

<u>Proof:</u>

Let
$$u = \sup\{f(x) + a : x \in X\},$$
$$v = \sup\{f(x) : x \in X\}.$$

We want to prove that $u = v + a$. Instead of trying to prove equality directly, we'll prove $u \leq v + a$ and $u \geq v + a$.

. <u>Show $u \leq v + a$</u>

Since v is an upper bound for $\{f(x) : x \in X\}$, we have

$$f(x) \leq v \quad \text{for all } x \in X.$$

Therefore $$f(x) + a \leq v + a \quad \text{for all } x \in X.$$

Hence $v + a$ is an upper bound for $\{f(x) + a : x \in X\}$.

But u is the <u>least</u> upper bound for this set, so $u \leq v + a$.

<u>Exercise:</u> Show $v + a \leq u$ & also prove the equality for inf's. $\blacksquare$

Exercises

1. A subset of a countable set is countable.

2. A superset of an uncountable set is countable.

3. $\exists$ injection $f: A \to \mathbb{N} \Rightarrow A$ is countable.

4. $\exists$ injection $f: \mathbb{R} \to A \Rightarrow A$ is uncountable.

Lemma

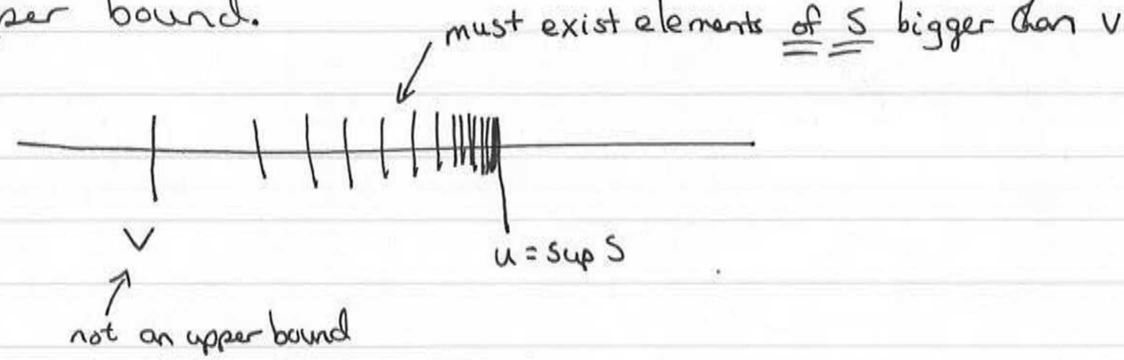
Given $S \subseteq \mathbb{R}$, $S \neq \emptyset$. Then

$$u = \sup S \iff \begin{array}{l} \text{(a) } \nexists\, s \in S \text{ such that } u < s \\ \text{(b) If } v < u \text{ then } \exists\, s \in S \text{ s.t. } v < s. \end{array}$$

Proof:

$\Rightarrow$ Suppose $u = \sup S$. Then, by definition, u is an upper bound for S, so $s \leq u$ for all $s \in S$. Hence (a) holds.

To prove (b), suppose that $v < u$. Then v is not an upper bound for S because u is the least upper bound.



Therefore, since v is not an upper bound, there must be some $s \in S$ such that $s > v$.

$\Leftarrow$ Exercise. ∎

<u>Exercise:</u> Show that if a set S has a supremum, then that supremum is unique.

This leaves open the question: does every set have a supremum? We take as an axiom that all bounded sets do have suprema.

<u>Axiom</u> <u>Supremum Property</u>

Every nonempty set $S \subseteq \mathbb{R}$ which has an upper bound has a supremum.

Some applications of suprema:

<u>Theorem</u> <u>Archimedean Property</u>

$\forall x \in \mathbb{R} \quad \exists n_x \in \mathbb{N}$ s.t. $x < n_x$.

<u>Proof:</u>
Suppose not. This means that $\exists$ at least one $x \in \mathbb{R}$ such that $(\exists n_x \in \mathbb{N},\ x < n_x)$ is false.

Therefore, for this x, we must have $x \geq n$ for all $n \in \mathbb{N}$!

Hence x is an upper bound for $\mathbb{N}$. By the supremum property, $\mathbb{N}$ then has a supremum u, & $u \leq x$.

But $u - 1 < u = \sup \mathbb{N}$, so by previous lemma $\exists n \in \mathbb{N}$ s.t. $u - 1 < n$. But then $u < n + 1$, contradicting that u is an upper bound for $\mathbb{N}$. $\blacksquare$

<u>Corollary</u> Let $y, z > 0$.

(a) $\exists\, n \in \mathbb{N}$ s.t. $ny > z$.

(b) $\exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < z$.

(c) $\exists\, n \in \mathbb{N}$ s.t. $n-1 \leq y < n$.

<u>Proof:</u>

(a) Let $x = z/y$. Then $\exists\, n \in \mathbb{N}$ s.t. $x < n$.

Hence $z = xy < ny$.

(b), (c) Exercises. ∎

<u>Theorem</u>: Existence of "$\sqrt{2}$".

$\exists x \in \mathbb{R}$ s.t. $x^2 = 2$.

<u>Proof:</u> (with motivations)
Let $S = \{y \in \mathbb{R} : 0 \leq y \ \&\ y^2 \leq 2\}$.

This set is bounded above, for example, 2 is an upper bound. Therefore it has a supremum x. Since S contains some positive numbers, e.g., $1 \in S$, we know that $x > 0$ (in fact, $x \geq 1$ at least since $1 \in S$).

We will show that $x^2 = 2$. We will do this by showing that $x^2 < 2$ and $x^2 > 2$ are impossible.

Suppose that $x^2 < 2$. Then we expect that $(x+y)^2$ should still be < 2 if y is small enough, and in particular we should have $(x + \frac{1}{n})^2 < 2$ if n is large enough. Let's try to prove that there is such an n. Note first that

$$\left(x+\frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$$

$$\leq x^2 + \frac{2x}{n} + \frac{1}{n}$$

idea:
an approximation here will be easier to work with

$$= x^2 + \frac{2x+1}{n}$$

only has n's & no n^2's.

To get this < 2 we need

$$x^2 + \frac{2x+1}{n} < 2$$

or

$$\frac{2x+1}{n} < 2 - x^2.$$

Why does such an n exist? Since $x^2 < 2$, we know that $0 < 2 - x^2$. Hence $0 < \frac{2-x^2}{2x+1}$.

Then by a previous corollary, there is some n such that $0 < \frac{1}{n} < \frac{2-x^2}{2x+1}$. This is the n we need.

So, we found an $n \in \mathbb{N}$ such that $\left(x+\frac{1}{n}\right)^2 < 2$.

But $x + \frac{1}{n} \in S$ by def. of S, contradicting the fact that x is an upper bound for S.

Thus $x^2 < 2$ is impossible.

Exercise: Show $x^2 > 2$ is impossible. $\blacksquare$

Thus we've shown $\sqrt{2}$ exists, & a previous theorem showed that if $\sqrt{2}$ existed, it could not be rational. Hence $\sqrt{2}$ must be irrational.

<u>Corollary</u> Given $z > 0$.

(a) $\exists$ rational r s.t. $0 < r < z$.

(b) $\exists$ irrational x s.t. $0 < x < z$.

<u>Proof:</u>

(a) We know $\exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < z$, & $\frac{1}{n}$ is rational.

(b) Since $\sqrt{2}, z > 0$, by a previous corollary $\exists n \in \mathbb{N}$ such that $\sqrt{2} < nz$. Hence $0 < \frac{\sqrt{2}}{n} < z$ for that n.

Exercise: $\sqrt{2}/n$ is irrational. $\blacksquare$

<u>Theorem</u> Given $x<y$.

(a) $\exists$ rational r s.t. $x<r<y$.

(b) $\exists$ irrational ξ s.t. $x<\xi<y$.

<u>Proof:</u> (Motivational picture on next page)

(a) $\exists\, m\in\mathbb{N}$ s.t. $0<\frac{1}{m}<y-x$.

Then $S=\{k\in\mathbb{N} : x<\frac{k}{m}\}$ has a least element, which we'll call n. Since n is the least element of this set, we must have

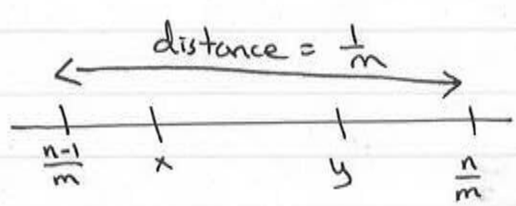
$$x<\frac{n}{m} \quad (\text{since } n\in S)$$

&

$$x\geq\frac{n-1}{m} \quad (\text{since } n-1\notin S).$$

Now, if $y\leq\frac{n}{m}$ then

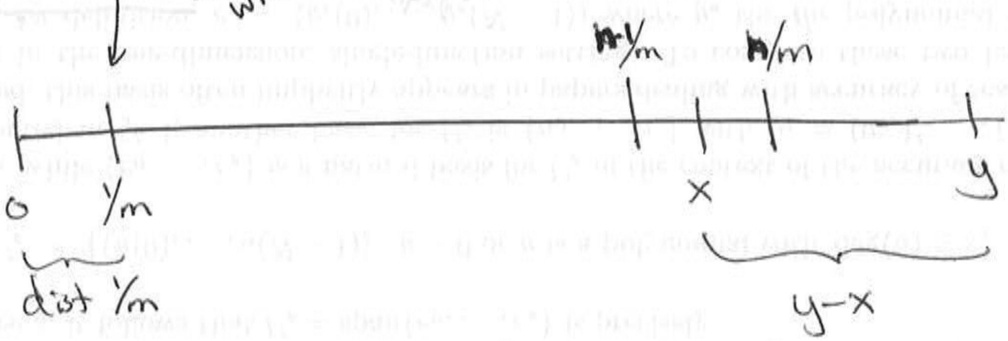
$$\frac{n-1}{m}\leq x<y\leq\frac{n}{m}$$



which implies $y-x\leq\frac{n}{m}-\frac{n-1}{m}=\frac{1}{m}$,

which is impossible. Therefore $x<\frac{n}{m}<y$.

Make this small enough so that some multiple will fall between x & y



Need $1/m < y - x$

Otherwise there may not be a multiple $\frac{n}{m}$ that falls between x & y.

(b) Choose any irrational $\xi > 0$, e.g., $\xi = \sqrt{2}$.

Then by part (a), $\exists$ rational r s.t.

$$\frac{x}{\xi} < r < \frac{y}{\xi}.$$

But then $x < r\xi < y$ & $r\xi$ is irrational (exercise). $\blacksquare$