5. Order Properties of $\mathbb{R}$

Axiom

$\exists P \subseteq \mathbb{R}$, $P \neq \emptyset$, s.t.

(a) $a, b \in P \Rightarrow a+b \in P$

(b) $a, b \in P \Rightarrow ab \in P$

(c) $a \in \mathbb{R} \Rightarrow$ exactly one of $a \in P$, $a=0$, or $-a \in P$.

Definition

P = set of strictly positive numbers. $(P = \mathbb{R}^+)$

Def

$a \in P \Rightarrow a$ is strictly positive. Write $a > 0$.

$a \in P$ or $a = 0 \Rightarrow a$ is positive (or nonnegative). $a \geq 0$.

$-a \in P \Rightarrow a$ is strictly negative

$-a \in P$ or $a = 0 \Rightarrow a$ is negative (or non positive)

Now DEFINE ORDER

Definition

$a < b$ if ~~[illegible]~~ $b - a > 0$, i.e. $b - a \in P$.

$a \leq b$ if $b - a \geq 0$, i.e. $b - a \in P \cup \{0\}$.

Theorem $a, b, c \in \mathbb{R}$.

(a) $a > b$ & $b > c \Rightarrow a > c$

(b) Exactly one holds: $a > b$, $a = b$, or $a < b$.

(c) $a \geq b$ & $b \geq a \Rightarrow a = b$.

Proof

(a) $a > b \Rightarrow a - b \in P$.
$b > c \Rightarrow b - c \in P$.
$\Rightarrow (a-b) + (b-c) \in P$
$\Rightarrow a - c \in P \Rightarrow a > c$. $\boxtimes$

(b), (c) exercises.

Theorem

(a) $a \in \mathbb{R} \setminus \{0\} \Rightarrow a^2 > 0$

(b) $1 > 0$ (i.e., $1 \in P$)

(c) $\mathbb{N} \subseteq P$.

Proof

(c) Induction. True for $n = 1$. etc. $\boxtimes$

Staircase analogy

Other familiar properties in book.

Absolute Value $|\cdot| : \mathbb{R} \to \mathbb{R}$

$$|a| = \begin{cases} a, & a \geq 0, \\ -a & a < 0. \end{cases}$$

Familiar properties.

Triangle Inequality

$$\big|\, |a| - |b| \,\big| \leq |a \pm b| \leq |a| + |b|.$$

Proof

Start from
$$-|a| \leq a \leq |a|$$
$$-|b| \leq b \leq |b|$$

So
$$-|a| - |b| \leq a + b \leq |a| + |b|$$

So $|a+b| \leq |a| + |b|$. Same for $a - b$.

Then
$$|a| = |a + b - b| \leq |a+b| + |b|$$

So
$$|a| - |b| \leq |a+b|.$$

Similarly
$$|b| - |a| \leq |a+b|$$

So $\big|\, |a| - |b| \,\big| \leq |a+b|$. Same for $a - b$. $\square$

Extend by induction to $|a_1 + \dots + a_n| \leq |a_1| + \dots + |a_n|$.