$\mathbb{R}$ is the simplest example of a <u>vector space</u>. ~~approximation of functions: What is the closest function?~~

4. Algebraic Properties of $\mathbb{R}$

Addition & Multiplication of real numbers are <u>binary operations</u> on $\mathbb{R}$

$+$ is a function from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$

$$+((a,b)) = a+b$$

just use this notation for simplicity

Ex: $+((2,3)) = 5 = 2+3$

Usual Properties hold

See p.28 of Bartle.

$a+b = b+a$

$(a+b)+c = a+(b+c)$

$\exists\, 0 \in \mathbb{R}$ s.t. $a+0=a$

$\forall a \in \mathbb{R}\ \exists\, (-a) \in \mathbb{R}$ s.t. $a+(-a)=0$ Define $a-b = a+(-b)$

$a\cdot b = b\cdot a$

$(a\cdot b)\cdot c = a\cdot(b\cdot c)$

$\exists\, 1 \in \mathbb{R}$ s.t. $a\cdot 1 = a$

$\forall a \in \mathbb{R}\setminus\{0\}\ \exists\, (\frac{1}{a}) \in \mathbb{R}$ s.t. $a\cdot(\frac{1}{a}) = 1$

$a\cdot(b+c) = a\cdot b + a\cdot c$

$(a+b)\cdot c = a\cdot c + b\cdot c$

Familiar consequences of these properties. $a+x=b$ is solvable

Lemma k^2 even $\Leftrightarrow$ k even

Rational numbers

$a/b = a \cdot (\frac{1}{b})$ $\qquad$ $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z},\ b \neq 0\}$

Theorem

$\nexists\, r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof

Suppose $r \in \mathbb{Q}$, $r = p/q$, has the property $(p/q)^2 = 2$.

Assume p, q have no common factors. (p/q is in lowest terms)

$p^2 = 2q^2$ So $2 \mid p^2$. p^2 is even. p is even.

$p = 2k. \Rightarrow$ $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2$

$\Rightarrow q^2$ even $\Rightarrow q$ even

Contradiction. $\blacksquare$

Still leaves the question

Q. Does $\exists\, r \in \mathbb{R}$ s.t. $r^2 = 2$?

How would you prove this?