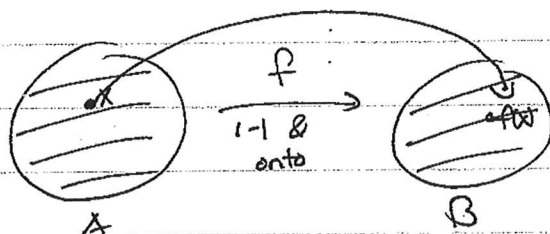


3. Finite & Infinite Sets



Every element of A lined up 1-1 with elements of B.

Definition

If $f: A \rightarrow B$ is a bijection, then we also say that f is a 1-1 correspondence, and that the sets A & B can be put into 1-1 correspondence with each other.

$D(f) = A$
 $R(f) = B$
 f is injective

Intuition: If \exists bijection $f: A \rightarrow B$ then A & B have the "same size" or the "same # of elements". We say they have the same cardinality.

Definition

$S_n = \{1, 2, \dots, n\}$ is an initial segment of \mathbb{N} .

(a) If \exists bijection $f: A \rightarrow S_n$ (or $f: S_n \rightarrow A$)

then A has n elements & is a finite set.

(b) If \exists bijection $f: A \rightarrow \mathbb{N}$ (or $f: \mathbb{N} \rightarrow A$)

then A is denumerable.

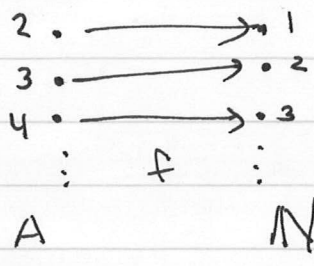
(c) A is countable if it is finite or denumerable.

(d) A is uncountable if it is not countable.

Homework: $m \neq n \Rightarrow \nexists$ bijection $f: S_m \rightarrow S_n$. (2)

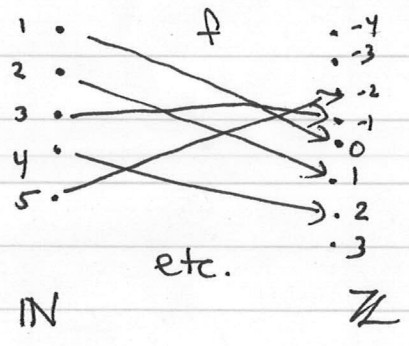
Examples

(a) $A = \{2, 3, 4, \dots\}$ is denumerable



$f(n) = n - 1$ is a bijection of A onto \mathbb{N} .

(b) \mathbb{Z} is denumerable

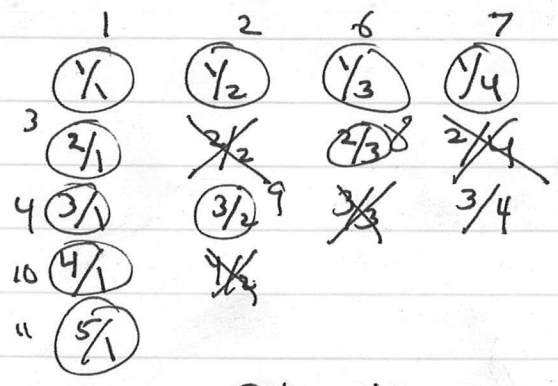


$f: \mathbb{N} \rightarrow \mathbb{Z}$ is a bijection.

Could give a formula for f if desired

$$f(n) = \begin{cases} n/2, & n \text{ even} \\ -(n-1)/2, & n \text{ odd} \end{cases}$$

(c) \mathbb{Q} is denumerable



$f(1) = 1/1$
 $f(2) = 1/2$
 $f(3) = 2/1$
 \vdots
 $f: \mathbb{N} \rightarrow \mathbb{Q}^+$
 f is a bijection

$$\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$$

(d) Any countable union of countable sets is ~~a~~ countable

Proof:

Suppose $S = \bigcup_{n=1}^{\infty} A_n$ with each A_n countable.

Let $A_n = \{a_{n1}, a_{n2}, \dots\}$

a_{11}	a_{12}	a_{13}	\dots
a_{21}	a_{22}	a_{23}	\dots
a_{31}	a_{32}	a_{33}	
\vdots			

Cross off duplicates & define \mathbb{R} bijection similarly as for \mathbb{Q}

Theorem \mathbb{R} is uncountable. (but \mathbb{Q} is "dense" in \mathbb{R} !)
Decimal expansions, nonuniqueness (just omit 0's & 9's)

Homework: $m \neq n \Rightarrow \nexists$ bijection $f: S_m \rightarrow S_n$.

Axiom: Well-ordering property of \mathbb{N}

Every ~~set~~ subset of \mathbb{N} contains a least element.

Axiom: Axiom of Choice (Weak Form)

Every infinite set contains a denumerable subset.

Examples A set can be in 1-1 correspondence with a proper subset of itself.

(a) $f: [0, 2] \rightarrow [0, 1]$ is a bijection
 $x \mapsto x/2$

(b) $f: (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ is a bijection
 $x \mapsto \tan x$

(c) Exercise: Find a bijection $f: [0, 1] \rightarrow (0, 1)$ Hint:

Can't be a continuous function

\mathbb{N} & \mathbb{R} are both infinite, yet they have different "sizes" (cardinality).

Completely nonintuitive since rationals (countable) & irrationals (uncountable) are both dense in \mathbb{R} !

Moreover, $\exists \infty$ many different cardinalities!

Example: Let $F = \{\text{all functions } f: \mathbb{R} \rightarrow \mathbb{R}\}$.

Then \nexists bijection $\alpha: \mathbb{R} \rightarrow F$.

Proof:

Suppose there was. $x \leftrightarrow f_x$.

Define $g(x) = f_x(x+1)$. Then $g \neq f_x$ for any $x \in \mathbb{R}$, contradiction. \square

Repeat: $\mathcal{F} = \{\text{functions } \alpha: F \rightarrow F\}$
 has cardinality $> F!$

Possible cardinalities: $1, 2, \dots$ (finite sets)
 $|\mathbb{N}|, |\mathbb{R}|, |F|, |F|, \dots$

Q. Are there only countably many cardinalities?

Continuum Hypothesis

There are no cardinalities between $|\mathbb{N}|$ & $|\mathbb{R}|$.

This is a set theory axiom: cannot be proved or disproved from the other axioms.

Either assume it is true & prove theorems, or
 assume it is false & prove theorems.
 (Doesn't matter for real analysis).

Other axioms we'll need:

Axiom: Well-ordering property of \mathbb{N} (not a set theory axiom)

Every subset of \mathbb{N} contains a least element.

Axiom: Axiom of Choice (weak form) (stronger form is a set theory axiom)

Every infinite set contains a denumerable subset.

Exercise: Find a denumerable subset of: \mathbb{R}
 \mathbb{F}
 \mathbb{J}

Q. Is $|\mathbb{R}| = |\mathbb{C}|$?

Yes. Space-filling curves.