21. Linear Functions

A function $f: \mathbb{R}^p \longrightarrow \mathbb{R}^q$ is <u>linear</u> if

$$\forall x, y \in \mathbb{R}^p, \quad \forall a, b \in \mathbb{R}, \quad f(ax+by) = af(x) + bf(y).$$

$\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) = \{ f: \mathbb{R}^p \to \mathbb{R}^q : f \text{ is linear} \}$

Suppose f is linear. Define

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \in \mathbb{R}^p$$

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Set

$$f(e_1) = a_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{q1} \end{bmatrix}, \quad \dots, \quad f(e_p) = a_p = \begin{bmatrix} a_{1p} \\ \vdots \\ a_{qp} \end{bmatrix} \quad \in \mathbb{R}^q.$$

Define

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_p \\ | & & | \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} \end{bmatrix}$$

$q \times p$ matrix.

Let $x = (x_1, \ldots, x_p) = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \in \mathbb{R}^p$.

Then

$$X = x_1 e_1 + \cdots + x_p e_p$$

So

$$f(x) = x_1 f(e_1) + \cdots + x_p f(e_p)$$

$$= x_1 a_1 + \cdots + x_p a_p$$

$$= \begin{bmatrix} a_{11} x_1 + \cdots + a_{1p} x_p \\ \vdots \\ a_{q1} x_1 + \cdots + a_{qp} x_p \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$$

$$= Ax.$$

ALL LINEAR FUNCTIONS ARE GIVEN BY MATRICES.

Define $$K = \max_{i,j} |a_{ij}|$$

Then

$$\|f(x)\| = \left(|a_{11}x_1 + \cdots + a_{1p}x_p|^2 + \cdots + |a_{q1}x_1 + \cdots + a_{qp}x_p|^2 \right)^{1/2}$$

$$\leq \left(\left(|a_{11}||x_1| + \cdots + |a_{1p}||x_p| \right)^2 + \cdots + \left(|a_{q1}||x_1| + \cdots + |a_{qp}||x_p| \right)^2 \right)^{1/2}$$

$$\leq \Big(\underbrace{(K|x_1| + \cdots + K|x_p|)^2 + \cdots + (K|x_1| + \cdots + K|x_p|)^2}_{q \text{ times}} \Big)^{1/2}$$

$$= \left(qK^2 (|x_1| + \cdots + |x_p|)^2 \right)^{1/2}$$

$$= q^{1/2} K \sum_{k=1}^{p} |x_k| \cdot 1$$

$$\leq q^{1/2} K \left(\sum_{k=1}^{p} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{p} 1^2 \right)^{1/2} \quad \text{Cauchy-Schwarz}$$

$$= q^{1/2} K \|x\| p^{1/2}$$

$$= (pq)^{1/2} K \|x\|.$$

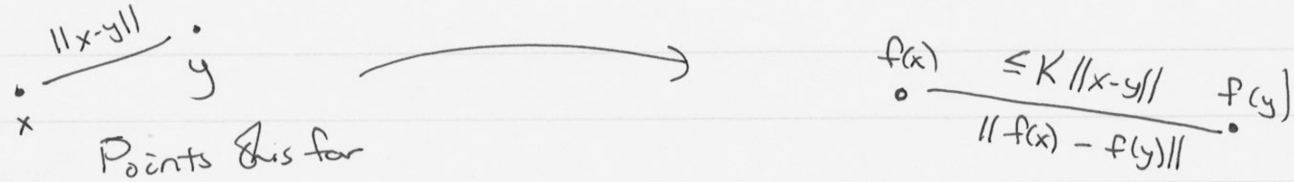
So:

$$\|f(x) - f(y)\| \overset{f \text{ linear}}{=} \|f(x-y)\| \leq (pq)^{1/2} K \|x-y\|$$

Definition

A function f is *Lipschitz* if $\exists K > 0$ s.t.

$$\forall x, y \in D(f), \qquad \|f(x) - f(y)\| \leq K \, \|x - y\|.$$



Points this far apart

map to points no more than K times farther apart

If $K < 1$ then we say f is *contractive*.

(In this case, $f(x)$ & $f(y)$ are always closer together than x & y were).

Note: We've shown that all *linear* functions are Lipschitz.

Example: $f(x) = |x|$ is Lipschitz but NOT linear.

(e.g., $f(-x) \neq -f(x)$)

<u>Theorem</u>

A Lipschitz function is continuous at all points in its domain.

<u>Proof:</u>

Suppose $x_n, a \in D(f)$ & $x_n \to a$. Then

$$\|f(x_n) - f(a)\| \leq K\|x_n - a\| \longrightarrow 0.$$

Hence $f(x_n) \longrightarrow f(a)$ so f is continuous at a. ∎

<u>Corollary</u>

All linear functions are continuous.

<u>Definition</u>

$f: \mathbb{R}^p \longrightarrow \mathbb{R}^q$ is Hölder continuous with exponent $\alpha > 0$ if

$\exists\, K > 0$ s.t.

$$\forall x, y \in \mathbb{R}^p, \qquad \|f(x) - f(y)\| \leq K \, \|x-y\|^{\alpha}.$$