20. Continuous Functions

Let f be a function mapping vectors in $\mathbb{R}^p$ to vectors in $\mathbb{R}^q$, with domain $D(f) \subseteq \mathbb{R}^p$ & range $R(f) \subseteq \mathbb{R}^q$.

Then <u>f is continuous at a point $a \in D(f)$</u> if:

$$\forall \text{ nbhd } V \text{ of } f(A) \quad \exists \text{ nbhd } U \text{ of } a \quad \text{s.t. } f(U) \subseteq V.$$

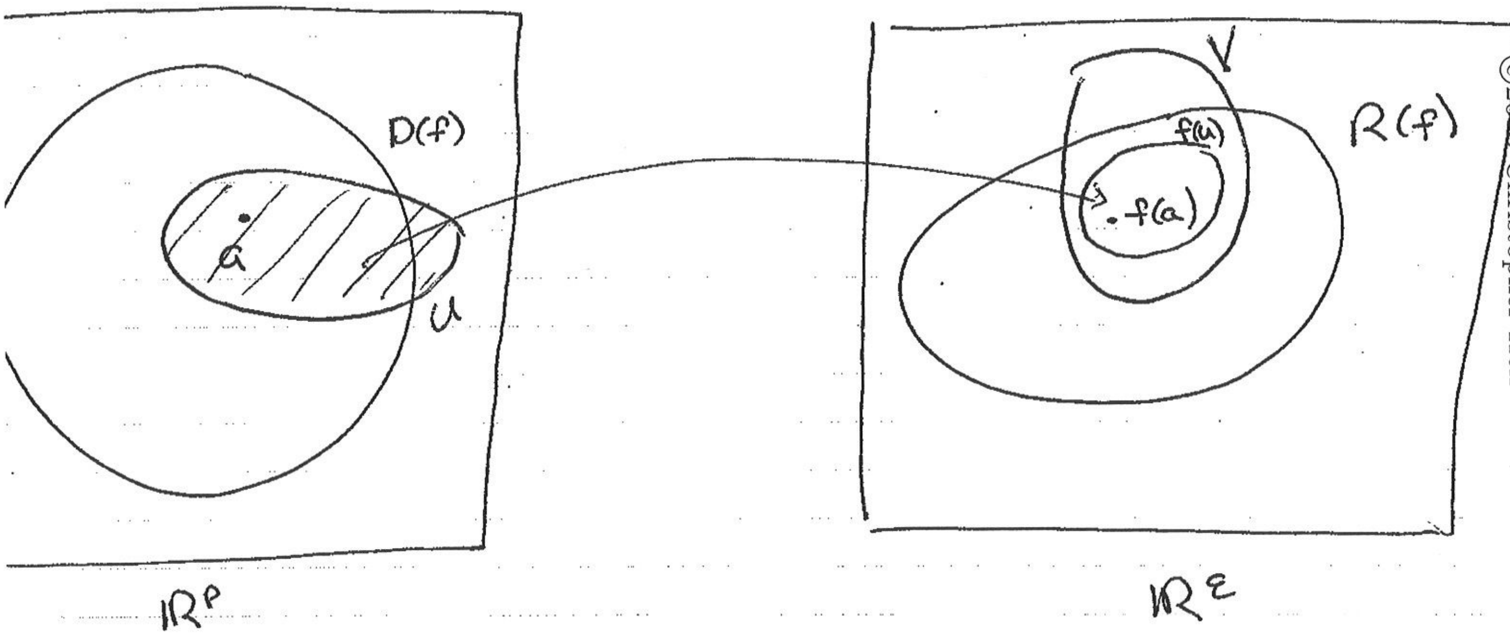

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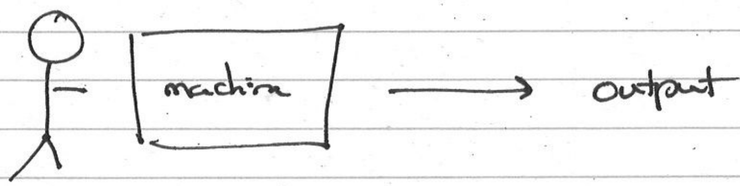
Illustration for Continuity

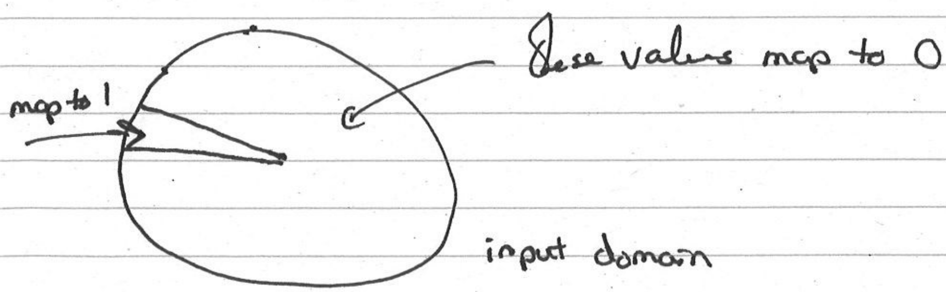


How well do you have to control the machine in order to achive a desired output?

To achieve .1 tolerance in output, what tolerance in input is needed?

If for any given output tolerance, there is an input tolerance that will achieve that, then the transformation from input to output is continuous.

If there is no input tolerance at all that will achieve a given output tolerance, the transformation is discontinuous.



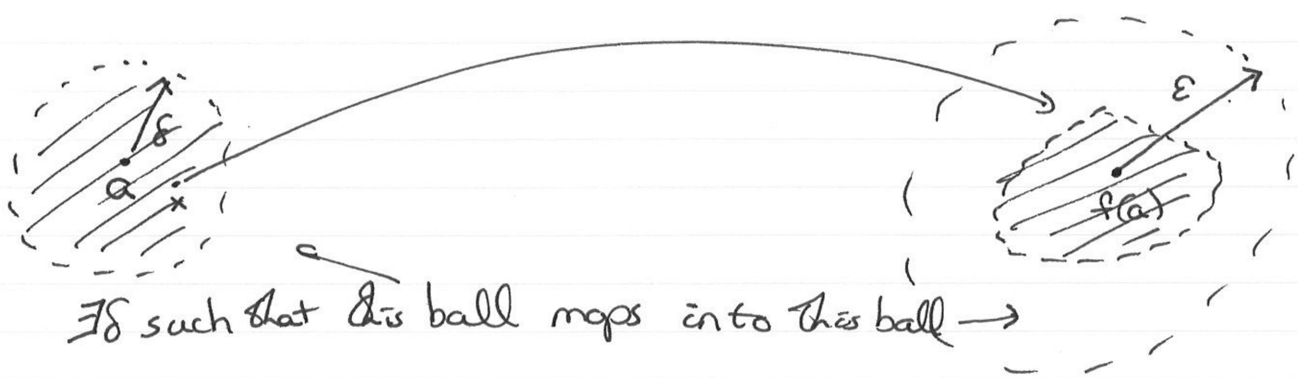
Theorem

Let $a \in D(f)$. Then the following statements are equivalent.

(a) f is continuous at (a).

(b) $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$$x \in D(f) \ \& \ \|x-a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

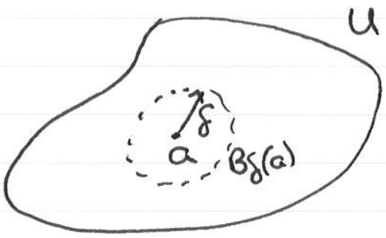


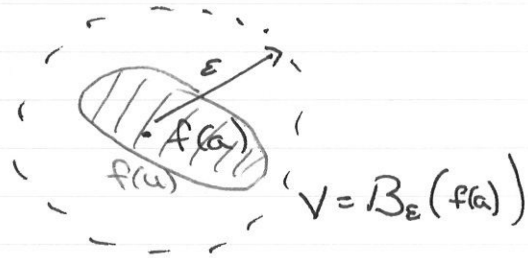
(c) $x_n, a \in D(f) \ \& \ x_n \to a \implies f(x_n) \to f(a)$

Proof of Theorem

(a) $\Rightarrow$ (b). Assume f is continuous at a.

Let $\varepsilon > 0$. Then $V = B_\varepsilon(f(a))$ is a neighborhood of $f(a)$





Hence $\exists$ nbhd U of a such that $f(U) \subseteq V$.

By def. of nbhd, $\exists \delta > 0$ s.t. $B_\delta(a) \subseteq U$.

Hence
$$f(B_\delta(a)) \subseteq f(U) \subseteq V = B_\varepsilon(f(a)).$$

That is,
$$\{f(x) : x \in D(f) \cap B_\delta(a)\} \subseteq B_\varepsilon(f(a)).$$

Thus,
$$\begin{aligned} x \in D(f) \ \& \ \|x - a\| < \delta &\Rightarrow x \in D(f) \cap B_\delta(a) \\ &\Rightarrow f(x) \in B_\varepsilon(f(a)) \\ &\Rightarrow \|f(x) - f(a)\| < \varepsilon, \end{aligned}$$

as desired.

(b) $\Rightarrow$ (c). Assume (b) holds.

Suppose $x_n \in D(f)$ & $x_n \to a$. We must show $f(x_n) \to f(a)$.
SHOW: $\forall \varepsilon > 0\ \exists N > 0$ s.t. $n \geq N \Rightarrow \|f(x_n) - f(a)\| < \varepsilon$.
So, choose $\varepsilon > 0$. By (b), $\exists\ \delta > 0$ s.t.

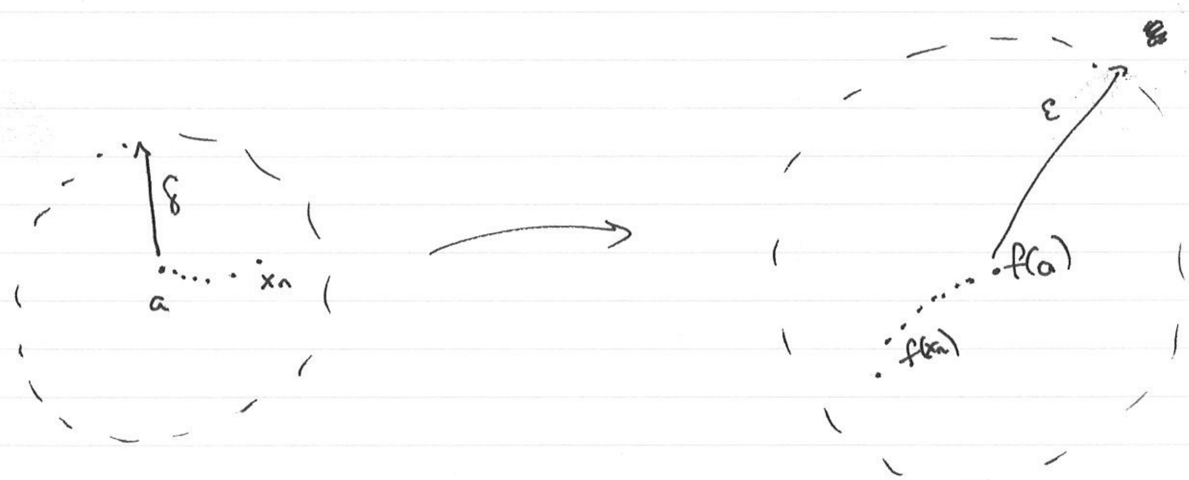
$$x \in D(f)\ \&\ \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

Now, $x_n \to a$, so by def. of limit $\exists N > 0$ s.t.

$$n \geq N \implies \|x_n - a\| < \delta.$$

Hence, if $n \geq N$ then $\|f(x_n) - f(a)\| < \varepsilon$.

Thus $f(x_n) \to f(a)$.



(c) $\Rightarrow$ (a). We'll prove the contrapositive statement, i.e.,
(a) false $\Rightarrow$ (c) false.

So, suppose (a) is false. Then,

$$\exists \text{ nbhd } V \text{ of } f(a) \text{ s.t. } \forall \text{ nbhd } U \text{ of } a, \quad f(U) \not\subseteq V.$$

Consider in particular the nbhd $U_n = B_{\frac{1}{n}}(a)$ of a.
We must have $f(U_n) \not\subseteq V$, or

$$\{f(x) : x \in D(f) \cap U_n\} \not\subseteq V.$$

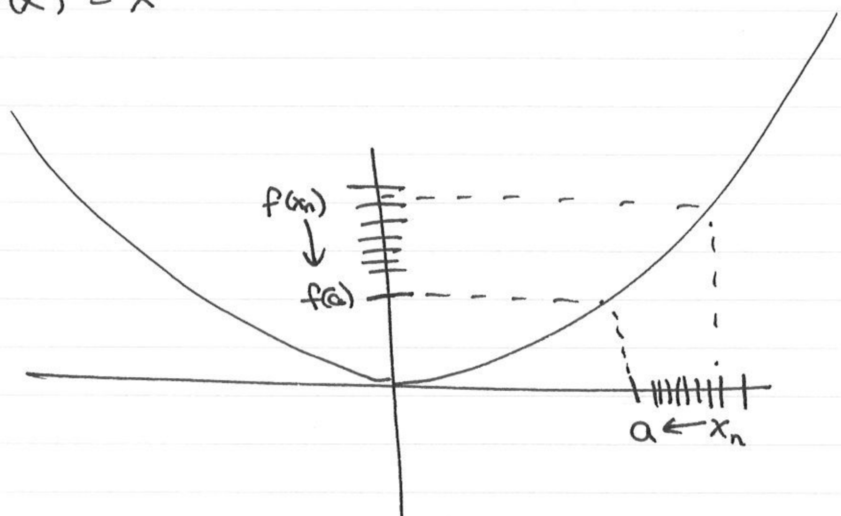
For this to happen, there must be some point
$x_n \in D(f) \cap U_n$ such that $f(x_n) \notin V$.
Then we have

$$x_n \in D(f) \;\&\; \|x_n - a\| < \tfrac{1}{n},$$

so $x_n \in D(f)$ & $x_n \longrightarrow a$. But $f(x_n) \notin V$ for any n.

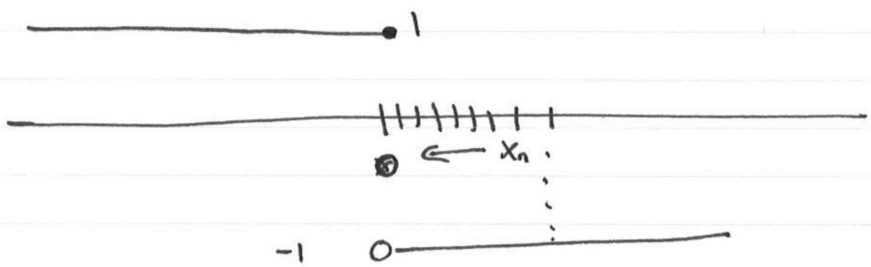
So $f(x_n) \not\to f(a)$. Therefore (c) is false, as desired. $\blacksquare$

Example $f(x) = x^2$



Continuous at all points

Example



$$f(x) = \begin{cases} 1, & x \le 0 \\ -1, & x > 0 \end{cases}$$

If $x_n > 0$ & $x_n \to 0$ then $f(x_n) = -1 \not\to 1 = f(0)$

Discontinuous at 0 (Continuous at other points!)

Example: $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

f is not continuous for <u>any</u> x.

$\exists\, r_n \longrightarrow x$ but $f(r_n) = 1 \quad \forall n$

$\exists\, \zeta_n \longrightarrow x$ $\quad f(\zeta_n) = 0 \quad \forall n$

So either $f(r_n) \not\to f(x)$

or $f(\zeta_n) \not\to f(x)$

Also: For $\varepsilon = 1/2$ $\nexists\, \delta$ that works.

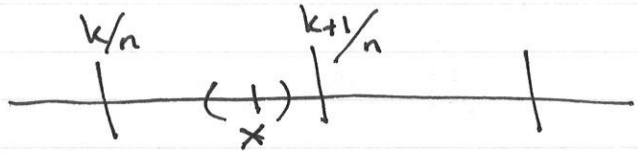
Example: $f(x) = \begin{cases} 1/n, & x = m/n \text{ in lowest terms} \\ 0, & x \text{ irrationals} \end{cases}$

is continuous <u>for irrational x only</u>.

Proof:

x is rational, $x = m/n$. $\quad \exists$ irrational $\zeta_k \to x \quad f(\zeta_k) = 0 \not\to \frac{1}{n} = g(x)$

x irrational. Choose $\varepsilon > 0$.



Rationals m/n are spaced a unit of $1/n$ apart.

Choose $1/n < \varepsilon$. Choose δ small enough that $(x-\delta, x+\delta)$

does not contain a rational of the form (for that n!)

$k/1, \ldots, k/n$ ← (this makes a closed set, dist is >0)

Suppose $y \in (x-\delta, x+\delta)$.

If y irrational then

$$|f(x) - f(y)| = |0-0| = 0 < \varepsilon.$$

If y is rational then $y = p/q$ with $q > n$, so

$$|f(x) - f(y)| = |0 - \tfrac{1}{q}| = \tfrac{1}{q} < \tfrac{1}{n} < \varepsilon. \quad \blacksquare$$

Exercises

~~[illegible]~~

If $f, g : \mathbb{R}^p \to \mathbb{R}^q$ are continuous at x, then so are

$$f+g, \quad f-g, \quad cf$$

If $f, g : \mathbb{R}^p \to \mathbb{R}$ are cont. at x then so are

$$fg, \qquad |f|$$

If $g(x) \neq 0$ then f/g

If $f : \mathbb{R}^p \to \mathbb{R}^q$ is cont. at a &

$g : \mathbb{R}^q \to \mathbb{R}^r$ " " " $b = f(a)$

then $g \circ f : \mathbb{R}^p \to \mathbb{R}^r$ " " " a