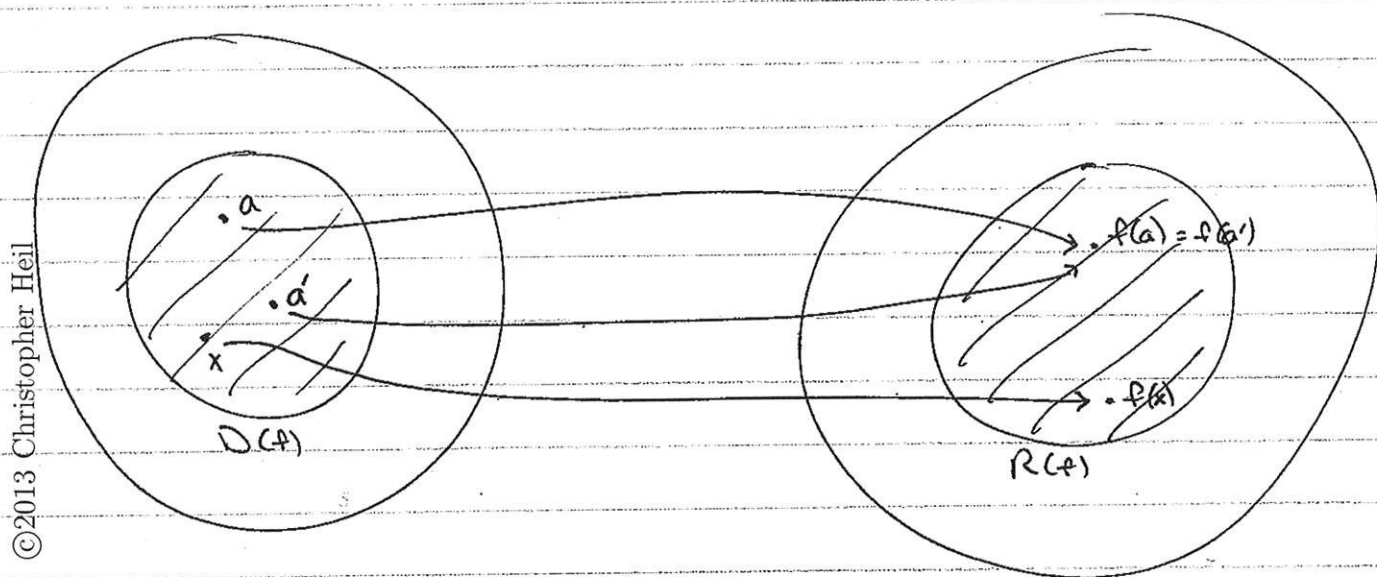


2. Functions

Informal Definition

A function f from a set A to a set B is a rule that assigns to each x in a fixed subset $D(f) \subseteq A$ a unique element $f(x) \in B$



Domain: $D(f) = \{a \in A : \text{there is some } b \in B \text{ s.t. } b = f(a)\}$

Range: $R(f) = \{f(a) : a \in A\}$

$= \{b \in B : b = f(a) \text{ for some } a \in A\}$

Sometimes write $x \mapsto f(x)$

\uparrow
"maps to"

Definition

If $D(f) = A$ we say f maps A into B and write

$f: A \rightarrow B$. \Leftarrow Note: This notation requires domain = A

If $R(f) = B$ we say f is surjective or onto.

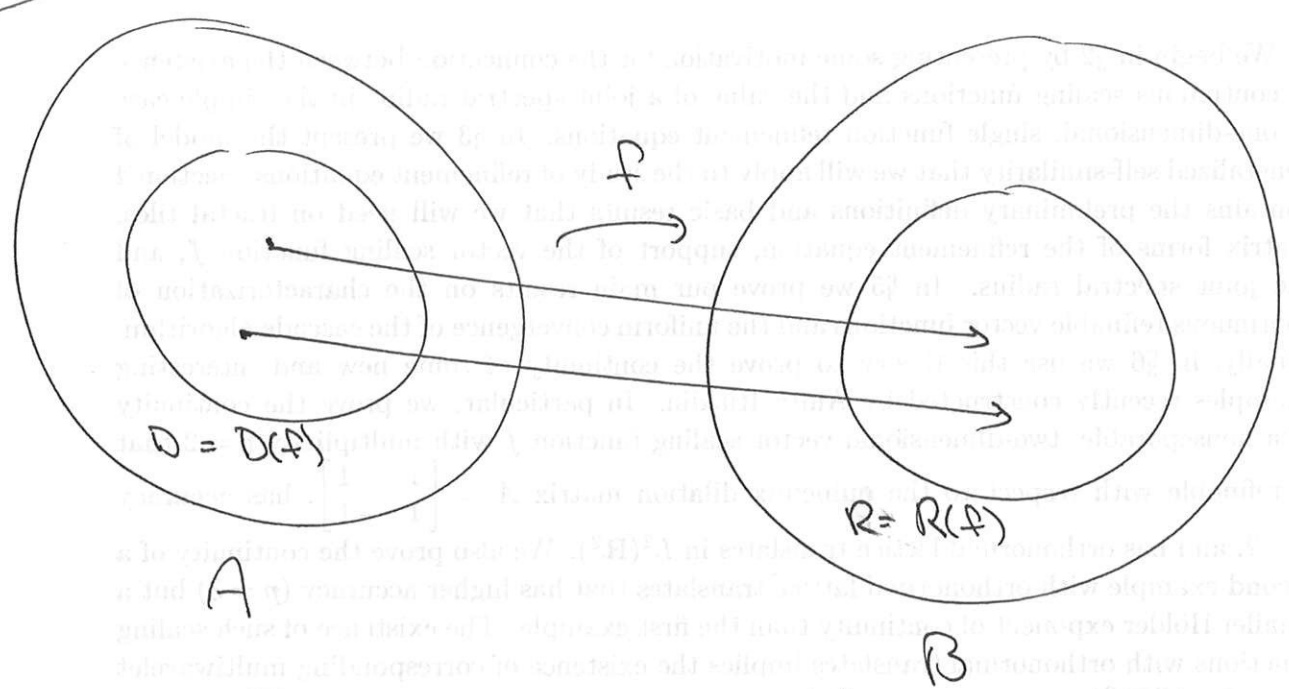
If $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$ then we say f is injective or 1-1.

Equivalent and more convenient:

f is injective if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

f is bijective, or a 1-1 correspondence, if it is both injective and surjective.

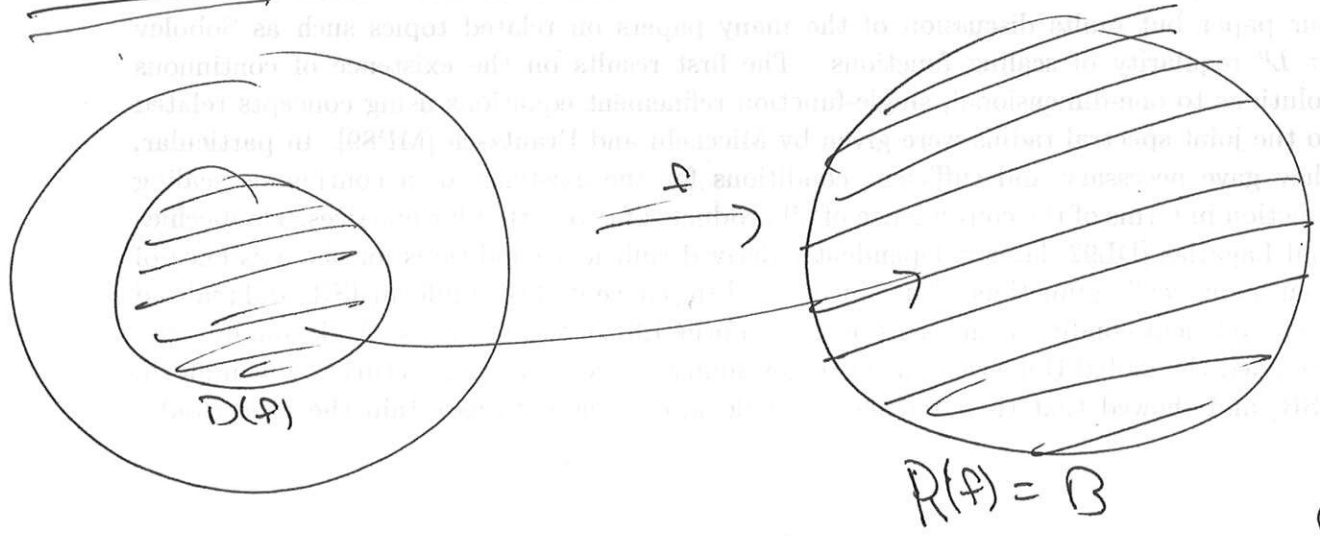
Injective (1-1)



Different points go different places.

OR: IF 2 points go to the same place, then they had to be the same point to begin with.

Surjective (Onto)



The range is everything.

Example $f(x) = \sqrt{x}$ $A = B = \mathbb{R}$ Injective
 Not surjective

If we change B to $B = [0, \infty)$ then f is surjective.

Example Let

$$F = \{ f : f \text{ is a function from } \mathbb{R} \text{ into } \mathbb{R} \}$$

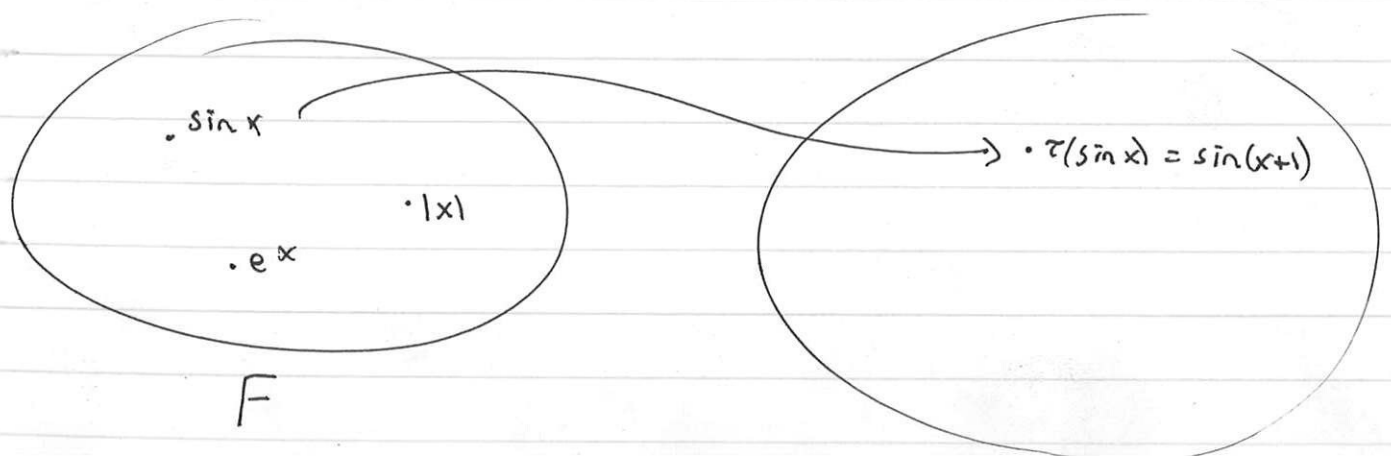
$$(f: \mathbb{R} \rightarrow \mathbb{R})$$

$f(x) = \sin x$ $g(x) = x^2$ $h(x) = |x|$ are all elements of F

Define $\tau: F \rightarrow F$ as follows.

For each $f \in F$, $\tau(f)$ is a new function $\tau(f): \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$(\tau(f))(x) = f(x+1)$$



EXERCISE: Show that τ is a bijection.

Injective Suppose $\tau(f) = \tau(g)$ for some $f, g \in F$.

This means that $\tau(f)$ & $\tau(g)$ are the same function, i.e., they define the same rule, so

$$(\tau(f))(x) = (\tau(g))(x) \text{ for all } x \in \mathbb{R}.$$

Hence $f(x+1) = g(x+1)$ for all $x \in \mathbb{R}$.

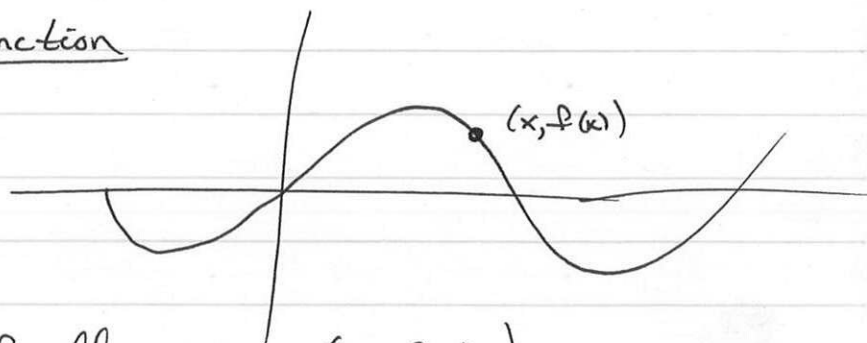
But this implies $f(t) = g(t)$ for every $t \in \mathbb{R}$,

so f & g define the same rule, hence are the same function, i.e., $f = g$.

Thus τ is injective.

Precise definition of a function

Motivation from graphs:



The graph is the set of all pairs $(x, f(x))$.

If you know f , you know the graph & conversely.

For an arbitrary setting, we define the function to be the graph.

Definition

Let A, B be sets. A function f from A to B is any set of ordered pairs $f \subseteq A \times B$ which has the property that

$$(a, b) \in f \text{ \& } (a, b') \in f \implies b = b'$$

Thus, ~~for~~ each a is associated with a unique b , ~~called~~ called the image of a & written

$$b = f(a).$$

$$D(f) = \{a \in A : (a, b) \in f \text{ for some } b \in B\}$$

$$R(f) = \{b \in B : (a, b) \in f \text{ for some } a \in A\}.$$

Example: $A = B = \mathbb{R}$

$$f = \{ (x, x^{1/2}) : x \geq 0 \}$$

is the precise meaning of saying "f is the function $f(x) = \sqrt{x}$ "

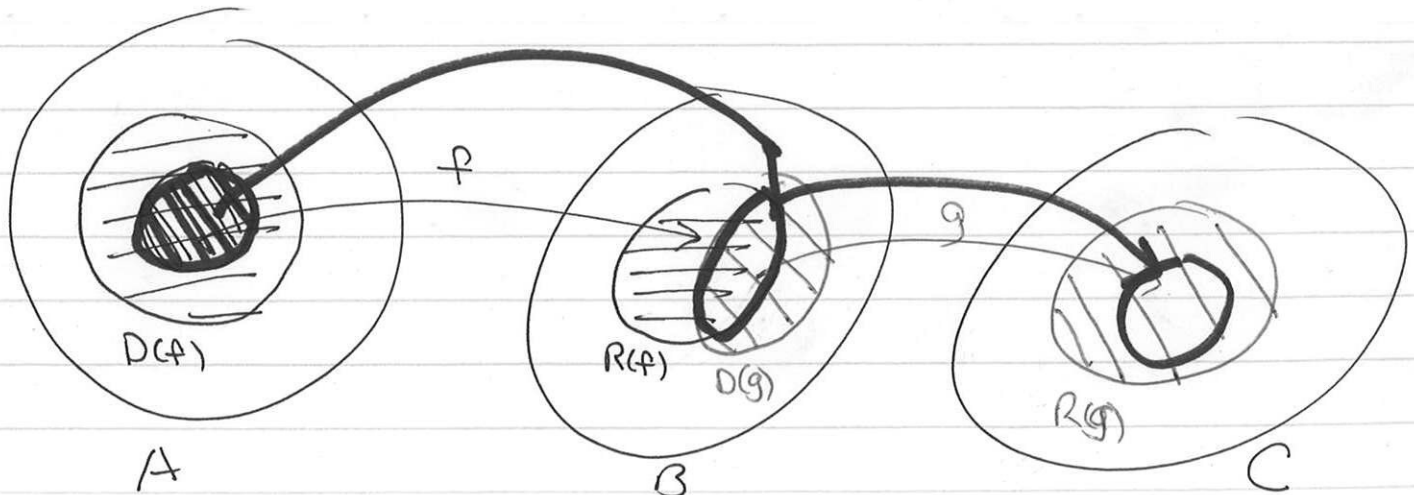
Definition Composition

~~Suppose~~ If f is a function from A to B
& g " " " " B to C

then ~~gof~~ ^{gof} is the function from A to C defined by the rule

$$\del{(g \circ f)(x)} \quad (g \circ f)(x) = g(f(x))$$

for all x's for which this makes sense



$$D(g \circ f) = \{ x \in D(f) : f(x) \in D(g) \}$$

$$R(g \circ f) = \{ g(f(x)) : x \in D(g \circ f) \}$$

Example Let $F = \{ \text{functions } f : \mathbb{R} \rightarrow \mathbb{R} \}$

Define for each $a \in \mathbb{R}$ a function $\tau_a : F \rightarrow F$ by

$$(\tau_a(f))(x) = f(x+a)$$

What is $\tau_a \circ \tau_b$?

Consider any $f \in F$. $(\tau_a \circ \tau_b)(f)$ is a function

$$\left((\tau_a \circ \tau_b)(f) \right)(x) = \left(\tau_a(\tau_b(f)) \right)(x)$$

$$= (\tau_b(f))(x+a)$$

$$= f((x+a)+b)$$

$$= f(x+(a+b))$$

$$= (\tau_{a+b}(f))(x)$$

Hence $(\tau_a \circ \tau_b)(f) = \tau_{a+b}(f)$. Since this is true for each f in the domain, $\tau_a \circ \tau_b$ & τ_{a+b} have the same rule, so are the same function, i.e. $\tau_a \circ \tau_b = \tau_{a+b}$.

Exercise

Define

$$\varphi: \mathbb{R} \longrightarrow F$$

$$a \longmapsto \tilde{a}$$

a. Is τ injective? Surjective?

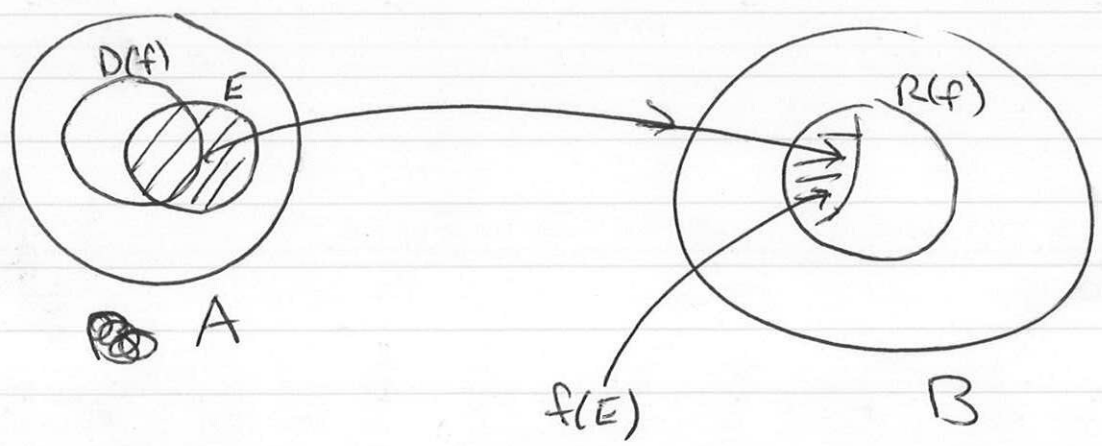
b. Show that φ respects the operations in \mathbb{R} & F :

$$\varphi(a+b) = \varphi(a) \circ \varphi(b), \quad a, b \in \mathbb{R}$$

Direct Image

The direct image of $E \subset A$ is

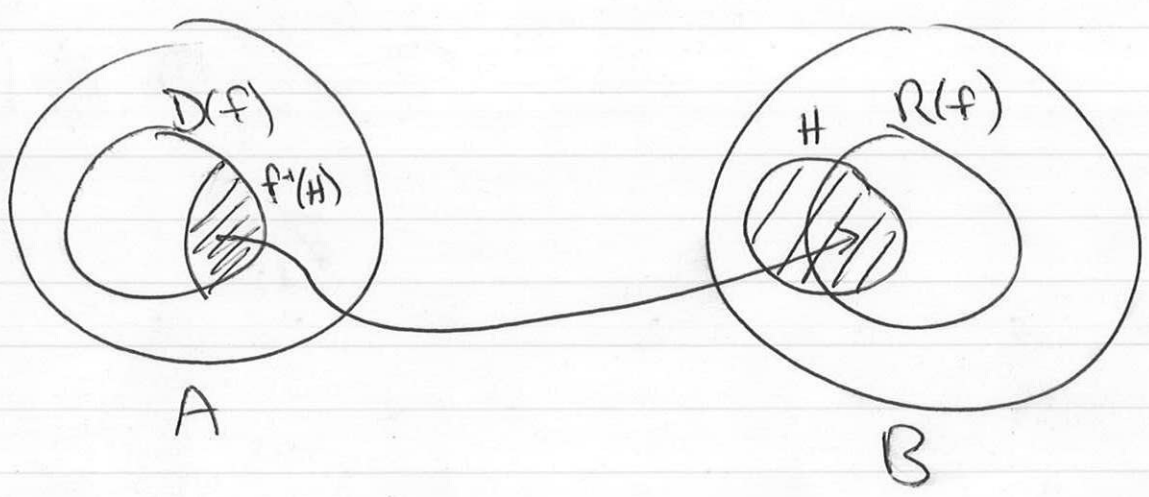
$$f(E) = \{f(x) : x \in E \cap D(f)\}$$



Inverse image

The inverse image of $H \subset B$ is

$$f^{-1}(H) = \{x \in D(f) : f(x) \in H\}$$



Note: Does NOT require f to be injective, is NOT the same as inverse function.

Example Length of a vector.

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R} \}$$

The length of $x \in \mathbb{R}^n$ is

$$l(x) = \|x\| = \sqrt{x_1^2 + \dots + x_n^2}$$

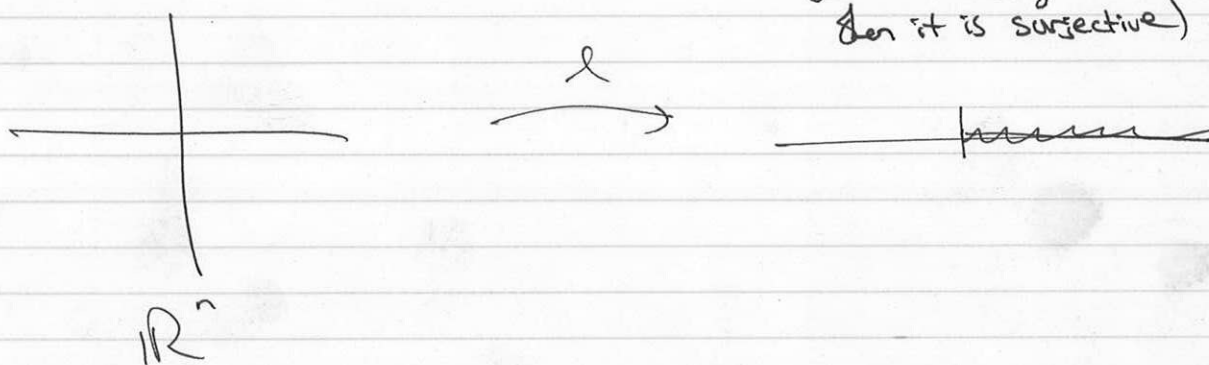
physical length or Euclidean length

Note l is a function:

$$l : \mathbb{R}^n \longrightarrow \mathbb{R}$$

l is not injective
not surjective

(^{but} if we change to $B = [0, \infty)$
then it is surjective)



$$D(l) = \mathbb{R}^n$$

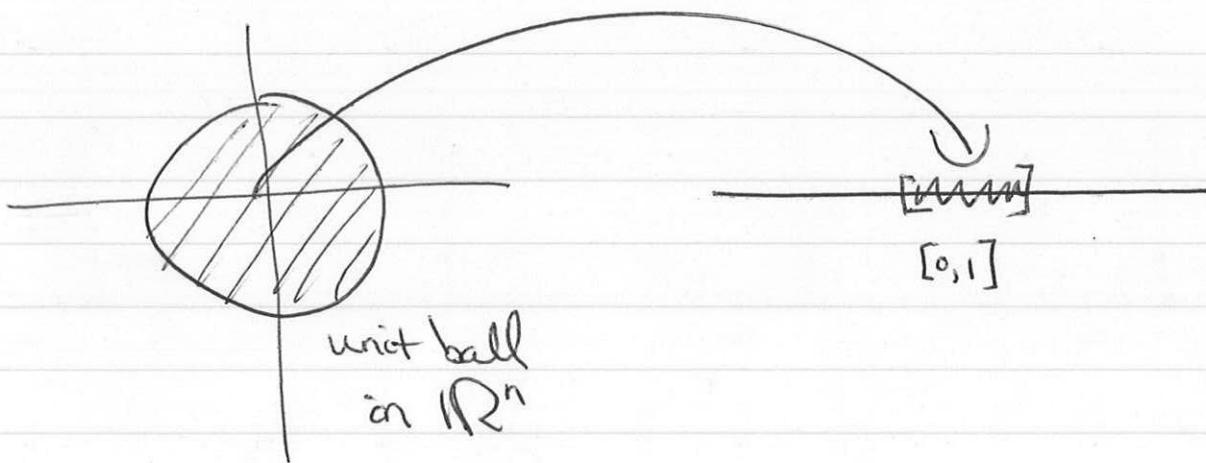
$$R(l) = [0, \infty)$$

l has special properties:

1. $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$
2. $\|x\| = 0 \iff x = 0$
3. $\|cx\| = |c| \|x\|$ for any scalar c
4. $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality).

study other functions
with these properties later

$$Q^{-1}([0,1]) = \{x \in \mathbb{R}^n : 0 \leq \|x\| \leq 1\}$$

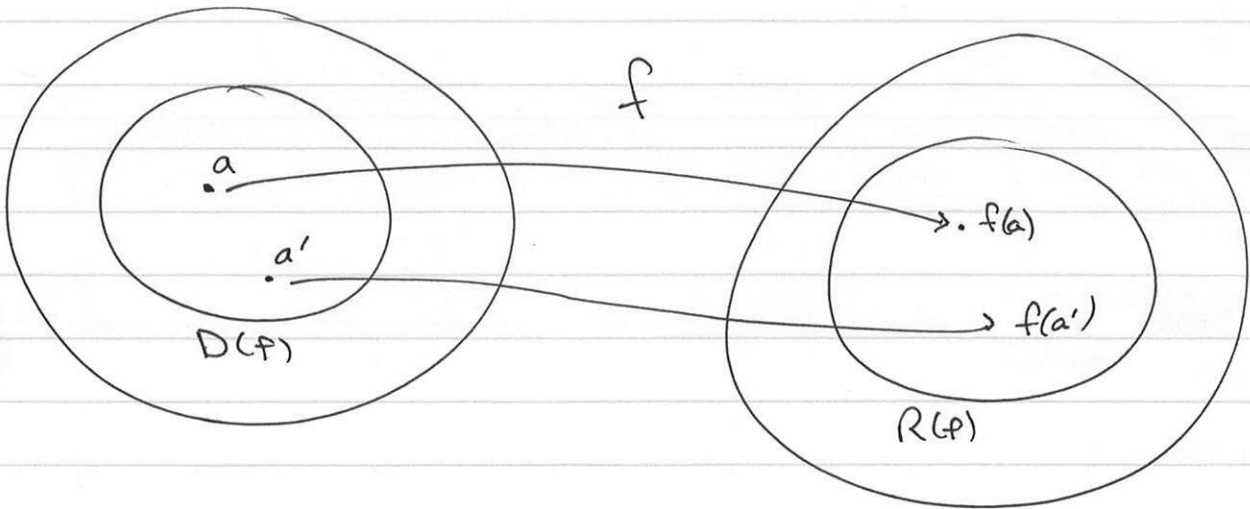


Note that

$$Q^{-1}([0,1]) = Q^{-1}([-1,1]) \quad \text{why?}$$

Inverse Function

Assume f is injective.



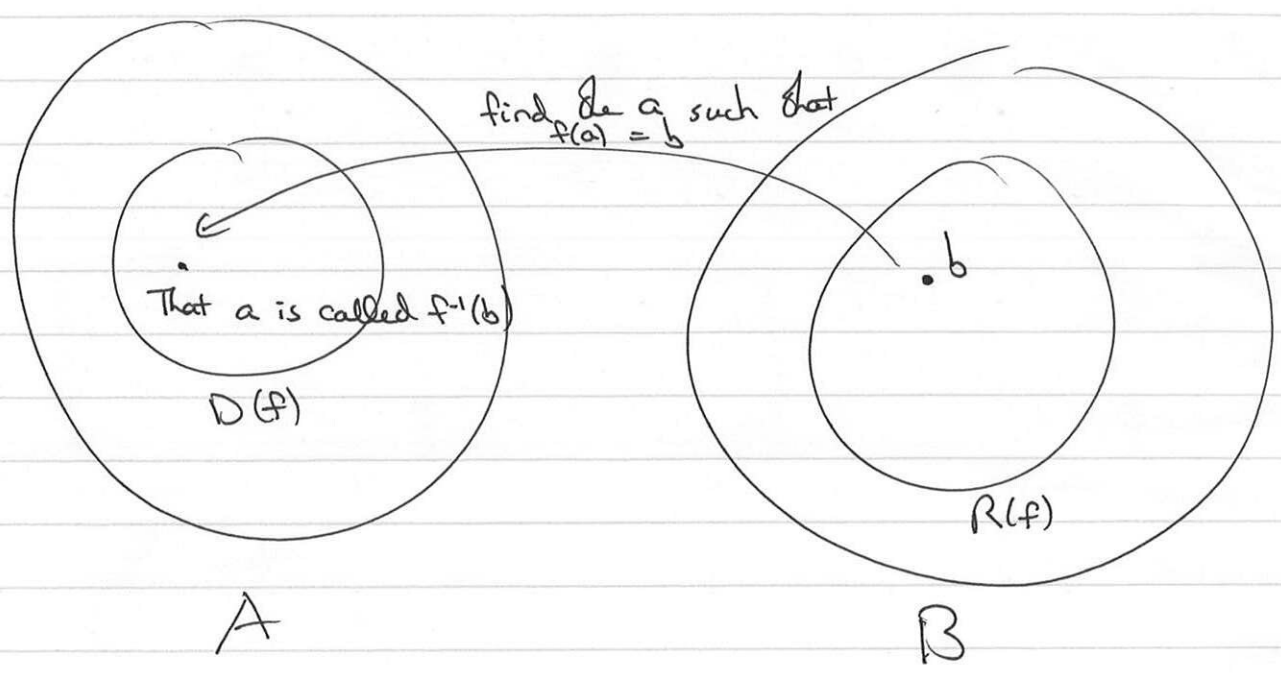
Each a is associated with a unique $f(a)$ in the sense that no other $a' \in D(f)$ has that same ~~image~~ image.

Thus each $b \in R(f)$ equals some ^{unique} $f(a)$.

Each $b \in R(f)$ has a unique $a \in D(f)$ s.t. $b = f(a)$.

Define f^{-1} from B to A by

$$f^{-1}(b) = a, \quad \text{the unique } a \text{ s.t. } b = f(a)$$



By def: $f^{-1}(f(a)) = a$

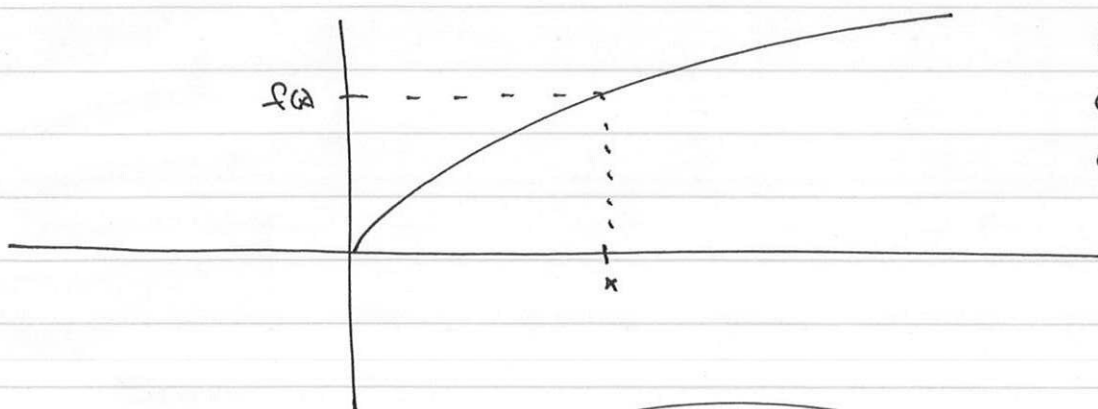
Because a is the unique element
s.t. $f(a) = f(a)$

Also: $f(f^{-1}(b)) = b$ because $a = f^{-1}(b)$ is
the unique element
s.t. $f(a) = b$

Exercise: $\tau_a: F \rightarrow F$ inverse function is $\tau_a^{-1} = \tau_a$

Ex: Calculus-type functions

$$f(x) = \sqrt{x} \quad A = \mathbb{R} \quad B = \mathbb{R}$$



Domain & range
combined in
one plot

