19. Some Extensions

Definition

Let (x_n) & (y_n) be sequences of real nos, $y_n \neq 0$ $\forall n$ large enough

a. (x_n) is <u>equivalent</u> to (y_n), written $(x_n) \sim (y_n)$, if

$$\lim_{n \to \infty} \frac{x_n}{y_n} = 1.$$

b. (x_n) is a <u>lower order of magnitude</u> than (y_n) if

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0$$ Write $x_n = o(y_n)$

c. (x_n) is <u>dominated by</u> (y_n) if $\left(\frac{x_n}{y_n}\right)$ is bounded:

$\exists A, B$ st. $A \le \frac{x_n}{y_n} \le B$ $\forall n$. Write $x_n = O(y_n)$.

Switching Order

Can't switch limits, or sums & limits, or integrals & limits, in general.

Need not be true that

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}$$

$$\sum_{m=1}^{\infty} \left(\lim_{n\to\infty} a_{m,n} \right) = \lim_{n\to\infty} \left(\sum_{m=1}^{\infty} a_{m,n} \right)$$

etc.

<u>Fatou for series</u> If $a_{mn} \geq 0 \ \forall m, n$ then

$$\sum_m \lim\overset{\text{inf}}{\cancel{\text{sup}}}_{n\to\infty} (a_{m,n}) \leq \lim\overset{\text{inf}}{\cancel{\text{sup}}}_{n\to\infty} \left(\sum_m a_{m,n}\right)$$

<u>Ex</u>:

$$\begin{array}{lccccl}
 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\
 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\
\text{Row } m & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \longrightarrow \lim\overset{\text{inf}}{\cancel{\text{sup}}}_{n\to\infty} a_{m,n} = 0 \longrightarrow \sum \lim\overset{\text{inf}}{\cancel{\text{sup}}} a_{m,n} = 0 \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

$\downarrow$

column n

$$\sum_m a_{m,n} = 2^{-n} \qquad \lim\overset{\text{inf}}{\cancel{\text{sup}}}\left(\sum_m a_{m,n}\right) = 0$$

<u>But</u>

$$\begin{array}{ccccl}
1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\
0 & 1 & \frac{1}{2} & \frac{1}{4} & \longrightarrow \lim\overset{\text{inf}}{\cancel{\text{sup}}}_{n\to\infty} a_{m,n} = 0 \qquad \sum_m \lim\overset{\text{inf}}{\cancel{\text{sup}}} a_{m,n} = 0 \\
0 & 0 & 1 & \frac{1}{2} & \\
 & & & 1 & \\
\vdots & & & &
\end{array}$$

$\downarrow$

$$\lim\overset{\text{inf}}{\cancel{\text{sup}}}_{n\to\infty}\left(\sum_m a_{m,n}\right) = 1$$

~~[illegible], etc.~~