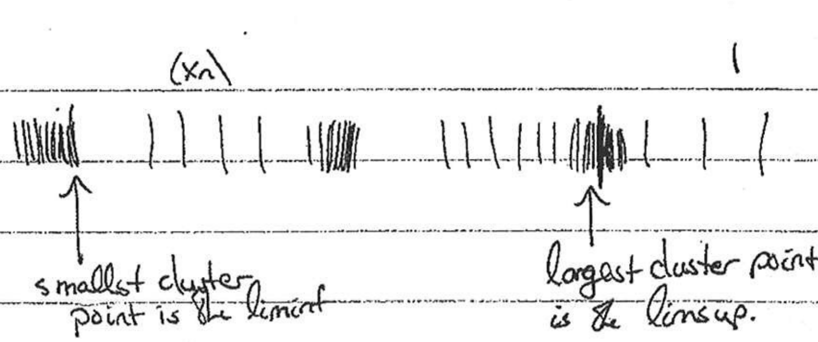
18. Limsups

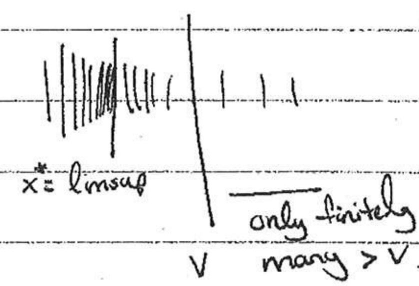
$\boxed{(x_n) \subseteq \mathbb{R}}$



Definition

$\limsup (x_n) = \limsup\limits_{n\to\infty} x_n$

$= \inf \{ v : \exists \text{ only finitely many } x_n > v \}$



$\liminf (x_n) = \sup \{ v : \exists \text{ only finitely many } x_n < v \}$

Note: (x_n) not bounded above $\Rightarrow$ no limsup (or limsup $= \infty$)

" " " below $\Rightarrow$ no liminf

All bounded sequences have a liminf & limsup.

UTILITY: $\liminf \le \lim \le \limsup$, $= \iff \lim$ exists.

$\uparrow$ may not exist

Lemma Let (x_n) be a bounded sequence in $\mathbb{R}$. Then TFAE:
Define $v_m = \sup\limits_{n \ge m} x_n$

conditions on the number are equivalent

(a) $x^* = \limsup (x_n)$

(b) $\forall \varepsilon > 0$, $\exists$ only finitely many $x_n > x^* + \varepsilon$, but ∞ many $x_n > x^* - \varepsilon$

(c) $x^* = \inf\limits_m v_m = \inf\limits_m \sup\limits_{n \ge m} x_n$.

(d) $x^* = \lim\limits_{m\to\infty} v_m = \lim\limits_{m\to\infty} \sup\limits_{n \ge m} x_n$.

(e) Let $S = \{ v : \exists (x_{n_k}) \text{ s.t. } x_{n_k} \to v \}$ Then $x^* = \sup (S)$

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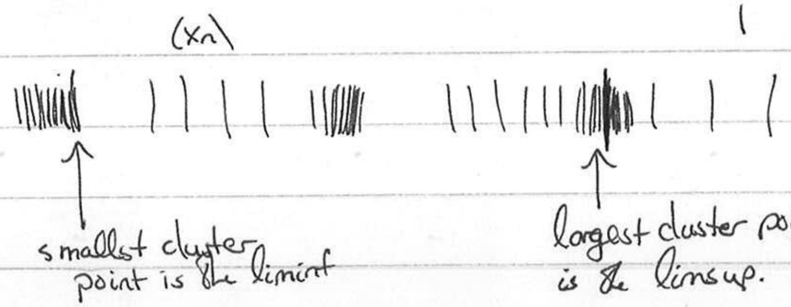

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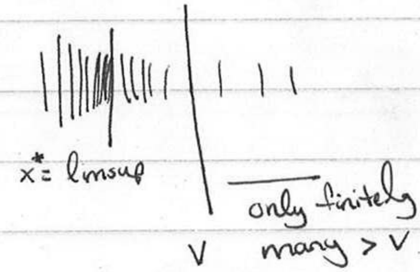

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All bounded sequences have a liminf & limsup.

UTILITY: liminf $\le$ lim $\le$ limsup, $= \Leftrightarrow$ lim exists.

lim: may not exist

Lemma Let (x_n) be a bounded sequence in $\mathbb{R}$. Then TFAE:

conditions on a number x^* are equivalent

Define $v_m = \sup\limits_{n\ge m} x_n$

(a) $x^* = \limsup(x_n)$

(b) $\forall \varepsilon > 0$, $\exists$ only finitely many $x_n > x^* + \varepsilon$, but ∞ many $x_n > x^* - \varepsilon$

(c) $x^* = \inf\limits_m v_m = \inf\limits_m \sup\limits_{n\ge m} x_n$.

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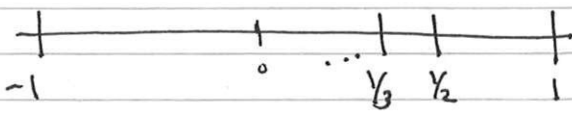
(e) Let $S = \{v : \exists (x_{n_k}) \text{ s.t. } x_{n_k} \to v\}$. Then $x^* = \sup(S)$.

<u>Example</u> $(x_n) = (1, -1, 1/2, -1, 1/3, -1, 1/4, -1, \dots)$

$V_1 = \sup_{n \geq 1} x_n = 1$

$V_2 = \sup_{n \geq 2} x_n = 1/2$



$V_3 = \qquad = 1/2$

$V_4 = \qquad = 1/3$

$\vdots \qquad \downarrow$ decreasing

$\inf V_n = 0 = \lim V_n$ so $\limsup x_n = 0$
$\liminf x_n = -1$

$\exists$ subsequences converging to -1 & to 0
largest of these is 0

Note that we <u>always</u> have

$$V_1 \geq V_2 \geq V_3 \geq \cdots$$

But: they could each be ∞

if some are finite, they could decrease to a finite value

or decrease forever

Proof:

(a) $\Rightarrow$ (b) Let $X^* = \limsup(x_n) = \inf\{v : \exists \text{ only fin. many } x_n > v\}$.

Suppose $\varepsilon > 0$ & there were only finitely many $x_n > X^* - \varepsilon$. Then $X^* - \varepsilon$ would be one of the v's, so the inf of all the v's would be $\leq X^* - \varepsilon$. But the inf is X^*, so this is impossible. Hence there are ∞ many $x_n > X^* - \varepsilon$.

Exercise: there are only fin. many $x_n > X^* + \varepsilon$.

(b) $\Rightarrow$ (c). Suppose (b) holds. Choose $\varepsilon > 0$.

Then $\exists$ only fin. many $x_n > x^* + \varepsilon$.

Thus $\exists$ N s.t. $x_n \leq X^* + \varepsilon \quad \forall n \geq N$.

Thus $$V_N = \inf_{n \geq N} x_n \leq X^* + \varepsilon.$$

So

$$\inf_m v_m \leq V_N \leq X^* + \varepsilon.$$

This is true $\forall \varepsilon > 0$, so $\inf_m v_m \leq X^*$.

Again choose $\varepsilon > 0$. $\exists\ \infty$ many $x_n \geq x^* - \varepsilon$.

So <u>no</u> <u>matter</u> what m is, there's an $n \geq m$ st. $x_n \geq x^* - \varepsilon$, s

$$v_m = \sup_{n \geq m} x_n \geq x^* - \varepsilon, \quad \forall m$$

Hence

$$\underbrace{\inf_m v_m}_{\text{glb of the } v_m} \geq \underbrace{x^* - \varepsilon}_{\text{a lower bound of the } v_m}$$

This is true $\forall \varepsilon$, so $\inf_m v_m = x^* = \limsup (x_n)$. ∎

(c) $\Rightarrow$ (d) Exercise — use the fact that the v_m are monotone decreasing.

(d) $\Rightarrow$ (e) Assume (d) holds. Let (x_{n_k}) be a convergent subsequence, say $x_{n_k} \longrightarrow v$. Note that $\forall m$,

$$v_m = \sup_{n \geq m} x_n \geq \sup_{n_k \geq m} x_{n_k} \geq \lim_{k \to \infty} x_{n_k} = v.$$

Hence

$$x^* = \lim_{m \to \infty} v_m \geq v.$$

Thus x^* is an upper bound for the set S.

On the other hand, we will show $x^* \in S$ – then $\sup(S) = x^*$.

To show $x^* \in S$ we must show $\exists$ subsequence $x_{n_k} \longrightarrow x^*$.

We know that $x^* = \lim_{m \to \infty} v_m$, and $v_m = \sup_{n \geq m} x_n$.

So:

$$v_1 = \sup_{n \geq 1} x_n \implies \exists n_1 \text{ st. } v_1 - 1 < x_{n_1} \leq v_1$$

$$v_{n_1+1} = \sup_{n \geq n_1+1} x_n \implies \exists n_2^{>n_1} \text{ st. } \underset{x^*}{\underset{\downarrow}{v_{n_1+1} - \tfrac{1}{2}}} < x_{n_2} \leq \underset{x^*}{\underset{\downarrow}{v_{n_1+1}}}$$

$$\vdots$$

Get (squeezing theorem)

$$x^* = \lim x_{n_k}$$

~~from $v_1 = \sup_{n \geq 1} x_n$, $\exists\, n_1$ s.t. $v_1 - \frac{1}{1} \leq x_{n_1} \leq v_1$.~~

~~Then " $v_2 = \sup_{n \geq 2} x_n$, $\exists\, n_2 > n_1$ s.t. $v_2 - \frac{1}{2} \leq x_{n_2} \leq v_2$~~

$\vdots$

~~etc. $v_k - \frac{1}{k} \leq x_{n_k} \leq v_k$~~

~~So $x^* = \lim_k (v_k - \frac{1}{k}) \leq \lim_k x_{n_k} \leq \lim_k v_k = x^*$.~~

~~Thus $x_{n_k} \to x^*$, so $x^* \in S$.~~

(e) $\Rightarrow$ (a). Assume (e) holds. Let $x^* = \sup(S)$.

We want to show that $x^* = \limsup(x_n) = \inf\{v : \exists \text{ only fin. many } x_n \geq v\}$.

Choose $\varepsilon > 0$. If $\exists\, \infty$ many $x_n > x^* + \varepsilon$ Then they would

have a convergent subsequence (x_{n_k}), say $x_{n_k} \to v$. Then $v \geq x^* + \varepsilon > x^*$.

But $v \in S$, so $\sup(S) \geq v > x^* = \sup(S)$, a contradiction.

Hence $\exists$ only fin. many $x_n > x^* + \varepsilon$. Hence

$x^* + \varepsilon$ is an upper bound for $\{v : \exists \text{ only fin many } x_n < v\}$.

So $x^* + \varepsilon \geq \limsup(x_n)$. True for all ε, so $x^* \geq \limsup(x_n)$.

On the other hand $x^* = \sup(S)$, so $\exists v \in S$ s.t.

$v \geq x^* - \frac{\varepsilon}{2}$. Hence $\exists (x_{n_k})$ s.t. $x_{n_k} \to v. \geq x^* - \frac{\varepsilon}{2} > x^* - \varepsilon$.

Therefore $\exists$ ∞ many $x_{n_k} > x^* - \varepsilon$.

$\limsup(x_n) = \inf\{v : \exists \text{ only fin. many } x_n > v\} \geq x^* - \varepsilon$.

True for all ε, so $\limsup(x_n) \geq x^*$. ∎

<u>Properties/Exercises</u>

(a) $\liminf(x_n) \leq \limsup(x_n)$

(b) $\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n)$
$\liminf(x_n + y_n) \geq \liminf(x_n) + \liminf(y_n)$
BUT NOT CONVERSE INEQ.!

(c) $x_n \leq y_n \ \forall n \Rightarrow$ $\liminf(x_n) \leq \liminf(y_n)$
$\limsup(x_n) \leq \limsup(y_n)$

$$\limsup(x_n + y_n) = \inf_m \left[\sup_{n \geq m} (x_n + y_n)\right] \leq \limsup x_n + \limsup y_n$$

<u>Example</u> $\liminf (x_n + y_n) \not\leq \liminf x_n + \liminf y_n$

$(x_n) = (1, -1, 1, -1, \dots)$ $\quad \liminf x_n = -1$

$(y_n) = (-1, 1, -1, 1, \dots)$ $\quad \liminf y_n = -1$

$(x_n + y_n) = (0, 0, \dots)$ $\quad \liminf (x_n + y_n) = 0$

(9)

MATH 4318 HOMEWORK #2: SOLUTIONS DUE: April 21, 1999

Work the following problems and hand in your solutions. A subset of these will be selected for grading.

1. (a) Use the definition

$$\limsup_{n\to\infty} x_n \;=\; \inf_m \sup_{n\geq m} x_n$$

to prove carefully that

$$\limsup_{n\to\infty} (a_n + b_n) \;\leq\; \limsup_{n\to\infty} a_n \;+\; \limsup_{n\to\infty} b_n.$$

Hint: Give names to things, e.g., define $u_m = \sup_{n\geq m} a_n$, $v_m = \sup_{n\geq m} b_n$, and $w_m = \sup_{n\geq m} (a_n + b_n)$.

Solution

In addition to the names above, define

$$u \;=\; \inf_m u_m, \qquad v \;=\; \inf_m v_m, \qquad w \;=\; \inf_m w_m.$$

We want to show that $w \leq u + v$.

We proved in the section on suprema that

$$w_m \;=\; \sup_{n\geq m} (a_n + b_n) \;\leq\; \sup_{n\geq m} a_n \;+\; \sup_{n\geq m} b_n \;=\; u_m + v_m.$$

Further, w is the inf of all the w_m, so we know that $w \leq w_m$ for every m. Hence $w \leq u_m + v_m$ for every m.

Let $\varepsilon > 0$ be given. Since the inf of the u_m is u, there exists some k such that $u_k < u + \varepsilon$. Since the u_m are decreasing, we conclude that $u_m \leq u_k < u + \varepsilon$ for all $m \geq k$. Similarly, there exists some ℓ such that $v_m \leq v_\ell < v + \varepsilon$ for all $m \geq \ell$. Let $j = \max\{k, \ell\}$. Then for $m \geq j$ we have BOTH $u_m < u + \varepsilon$ and $v_m < v + \varepsilon$. Therefore, for $m \geq j$,

$$w \;\leq\; u_m + v_m \;<\; (u + \varepsilon) + (v + \varepsilon) \;=\; u + v + 2\varepsilon.$$

Since this is true for EVERY $\varepsilon > 0$, we conclude that $w \leq u + v$. $\square$

(b) Either prove that the inequality in part (a) is an equality, or find a counterexample.

Solution

Let $(a_n) = (1, -1, 1, -1, 1, -1, \dots)$ and $(b_n) = (-1, 1, -1, 1, -1, \dots)$. Then $(a_n + b_n) = (0, 0, 0, \dots)$, so

$$\limsup_{n\to\infty} (a_n + b_n) \;=\; 0 \;<\; 2 \;=\; 1 + 1 \;=\; \limsup_{n\to\infty} a_n \;+\; \limsup_{n\to\infty} b_n. \quad \square$$

Theorem

Let $(x_n) \subseteq \mathbb{R}$ be bounded. Then:

$$(x_n) \text{ converges} \iff \liminf x_n = \limsup x_n.$$

In this case, this is the limit.

Proof:

$\Rightarrow$. Assume $x_n \to x$. Choose ε. $\exists N$ s.t.

$$n \geq N \implies x - \varepsilon \leq x_n \leq x + \varepsilon.$$

Hence $x - \varepsilon = \liminf (x - \varepsilon) \leq \liminf (x_n) \leq \limsup (x_n) \leq \limsup (x + \varepsilon) = x + \varepsilon.$

True for all ε, so $x \leq \liminf (x_n) \leq \limsup (x_n) \leq x$.

$\Leftarrow$ Assume $\liminf x_n = \limsup x_n = x$. Choose $\varepsilon > 0$.

Then $\exists$ only finitely many $x_n > x + \varepsilon$. Hence $\exists N_1$ st.

$$n \geq N_1 \implies x_n \leq x + \varepsilon.$$

Similarly $\exists N_2$ s.t. $n \geq N_2 \implies x_n \geq x - \varepsilon$ (look at liminf).

So $n \geq \max\{N_1, N_2\} \implies |x - x_n| \leq \varepsilon \implies x_n \to x.$ ∎

Infinite limsups

Suppose (x_n) is unbounded. Given $R>0$, if there were only finitely many $x_n > R$ then (x_n) would be bounded. Hence $\exists$ ∞ many $x_n > R$. Therefore, no matter what m is, $\exists\, n \geq m$ s.t. $x_n > R$. Hence

$$v_m = \sup_{n \geq m} x_n > R, \quad \text{all } m$$

so

$$\limsup_{n\to\infty} x_n = \inf_m v_m \geq R.$$

But R is arbitrary, so we write

$$\limsup_{n\to\infty} x_n = \infty.$$

Infinite limits

Given (x_n), we say (x_n) diverges to ∞ if

$$\forall R>0 \;\exists N>0 \text{ s.t. } \quad n \geq N \Rightarrow x_n > R.$$

In this case we write $\lim_{n\to\infty} x_n = \infty$

Note that x_n is not "converging to ∞"! We do not get "$|x_n - \infty| < \varepsilon$"