Exercise: Show this implies that

$$f_n \to g \text{ in } L^2\text{-norm} \implies f_n \to g \text{ in } L^1\text{-norm}.$$

However, $f_n \to g$ in L^1-norm $\not\Rightarrow$ $f_n \to g$ in L^2-norm (in general). Try

$$f_n(x) = \begin{cases} \frac{1}{n} x^{-1/2}, & 0 < x \le 1, \\ 0, & x = 0. \end{cases}$$

Then $f_n \to 0$ in L^1-norm but $f_n \not\to 0$ in L^2-norm.

Further, on an infinite domain, there is <u>no</u> relation between L^1 & L^2 norm in general.

For example, on the domain $(1, \infty)$, consider

$$f_n(x) = \frac{1}{x}, \qquad x \ge 1.$$

L^∞-norm = Uniform Norm

The L^∞-norm for piecewise continuous functions is also called the uniform norm, & is defined by

$$\|f\|_\infty = \sup_{x \in D} |f(x)| \qquad (*)$$

where D is the domain of f.

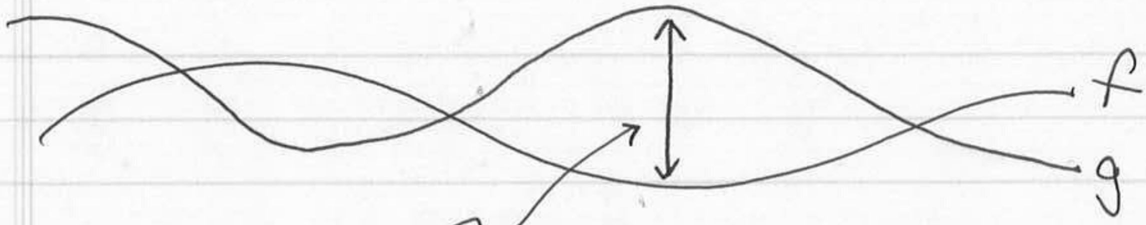
Note: For completely arbitrary functions, there is a subtle difference between the L^∞-norm and the uniform norm or sup-norm given by $(*)$.

That distinction is covered in a course on measure theory.

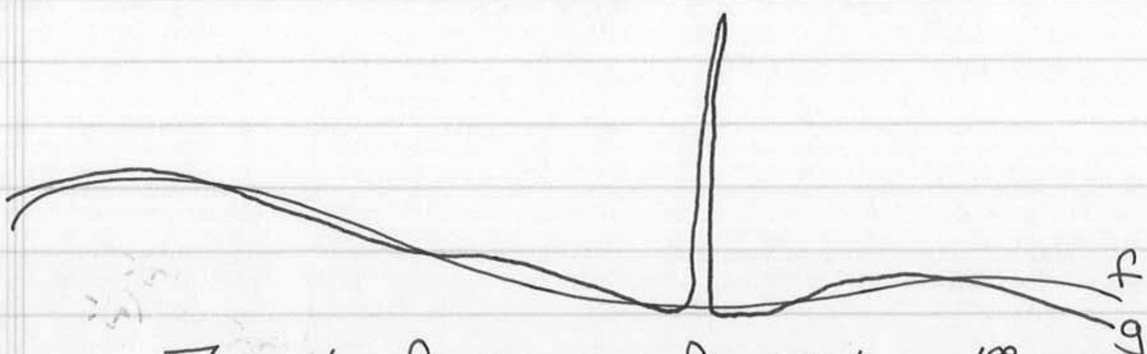
We say that

$f_n \to g$ in L^∞-norm or uniformly if

$$\lim_{n\to\infty} \|g - f_n\|_\infty = \lim_{n\to\infty} \Bigg(\underbrace{\sup_{x\in D} |g(x) - f_n(x)|}_{\text{This is the "maximum deviation" between } f \,\&\, g.} \Bigg) = 0.$$



The "maximum deviation" between f & g is $\|f-g\|_\infty$



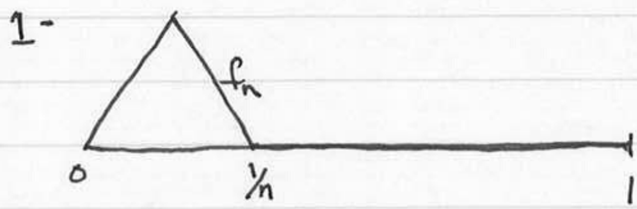
These two functions are far apart in L^∞-norm; even though $f(x)$ & $g(x)$ are close for most x's, there's an x where $|f(x)-g(x)|$ is large.

$\|f-g\|_\infty$ is <u>large</u>.

On the other hand, the <u>area</u> between the two graphs is small, so

$$\|f-g\|_1 = \int |f(x)-g(x)|\,dx \text{ is } \underline{\text{small}}.$$

<u>Example:</u>



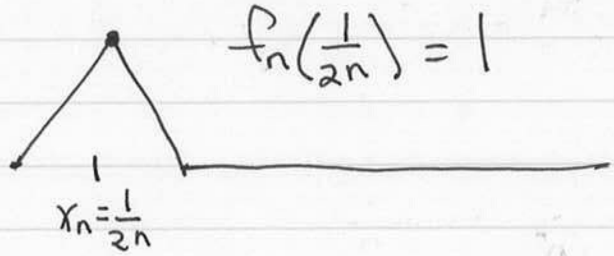
We saw before that $f(x)$ converges <u>pointwise</u> to the zero function:

$$\forall x \in [0,1], \quad \lim_{n\to\infty} |f_n(x) - 0| = 0$$

For each <u>individual</u> x, $f_n(x)$ eventually becomes zero.

However, there is always <u>some</u> x_n (depending on n!) where $f_n(x_n)$ & 0 are far apart

$$f_n\left(\frac{1}{2n}\right) = 1$$



The L^∞- distance between f_n & the zero function is

$$\|f_n - 0\|_\infty = \sup_{x\in[0,1]} |f_n(x) - 0| = 1$$

Thus $\|f_n - 0\|_\infty = 1 \quad \forall n$

So $f_n \not\to 0$ in L^∞-norm

or $f_n \not\to 0$ uniformly!

Thus:

$$\text{pointwise convergence} \not\Rightarrow \text{uniform convergence}$$

<u>Exercise</u> Show that

$$\text{uniform convergence} \Longrightarrow \text{pointwise convergence}$$

<u>Exercise</u>

Show that on a <u>finite domain</u> D,

$$\text{uniform convergence} \Longrightarrow L^1 \text{ convergence.}$$

But show that on an infinite domain (such as $D = \mathbb{R}$),

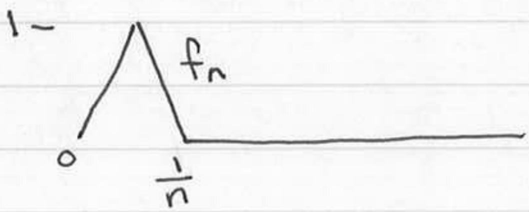
$$\text{uniform convergence} \not\Rightarrow L^1 \text{ convergence}$$

Hint: $f_n(x) = \frac{1}{n}$

Example/Exercise Show

$$L^1 \text{ convergence } \not\Longrightarrow \text{ uniform convergence}$$

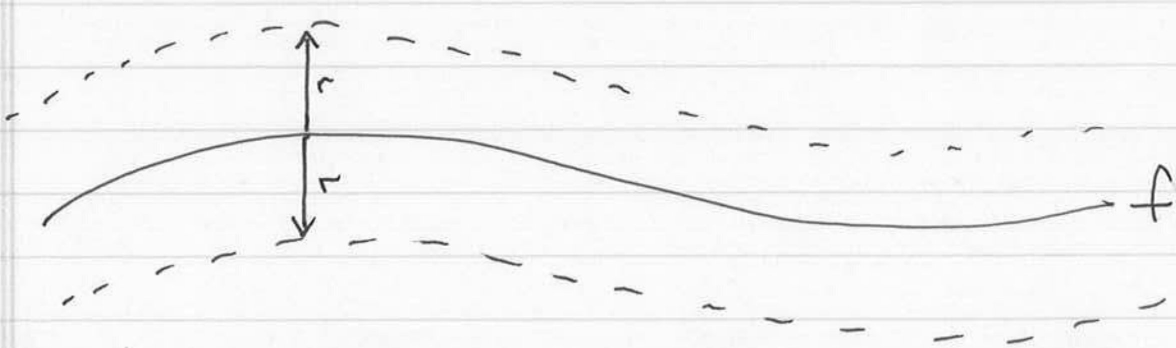
Consider



Picture for L^∞-norm

$$\|f-g\|_\infty = r \implies \sup_x |f(x)-g(x)| = r$$

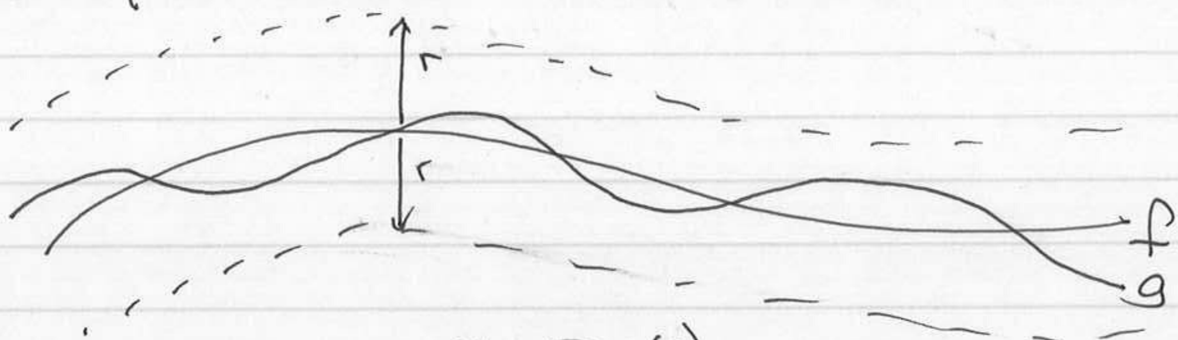
$$\implies |f(x)-g(x)| \leq r \quad \forall x$$



If $\|f-g\|_\infty < r$, then g must always take values between the dotted lines.

So: $g \in B_r(f) \iff \|f-g\|_\infty < r$

Any function g that lies between the dotted lines belongs to the ball of radius r centered at f. (with respect to the L^∞-norm!)



$g \in B_r(f)$

Bartle Notation

Instead of just considering functions that map numbers to numbers, we can consider functions that map vectors to vectors.

If $f: \mathbb{R}^p \to \mathbb{R}^q$, the analog of the L^∞-norm is

$$\|f\| = \sup_{x \in \mathbb{R}^p} \underbrace{\|f(x)\|}_{\text{norm of } f(x) \in \mathbb{R}^q}$$

(or could have f defined on a domain $I \subseteq \mathbb{R}^p$ instead of all of $\mathbb{R}^p$).

Bartle defines

$$\mathcal{B}_{pq}(I) = \left\{ f : I \to \mathbb{R}^q \ : \ \|f\| = \sup_{x \in I} \|f(x)\| < \infty \right\}$$

Gibbs' Phenomenon

Set

$$f_n(x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin((2k-1)2\pi x)}{2k-1}$$

$$g(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ 0 & x = 0, \frac{1}{2}, 1 \\ -1 & \frac{1}{2} < x < 1 \end{cases}$$

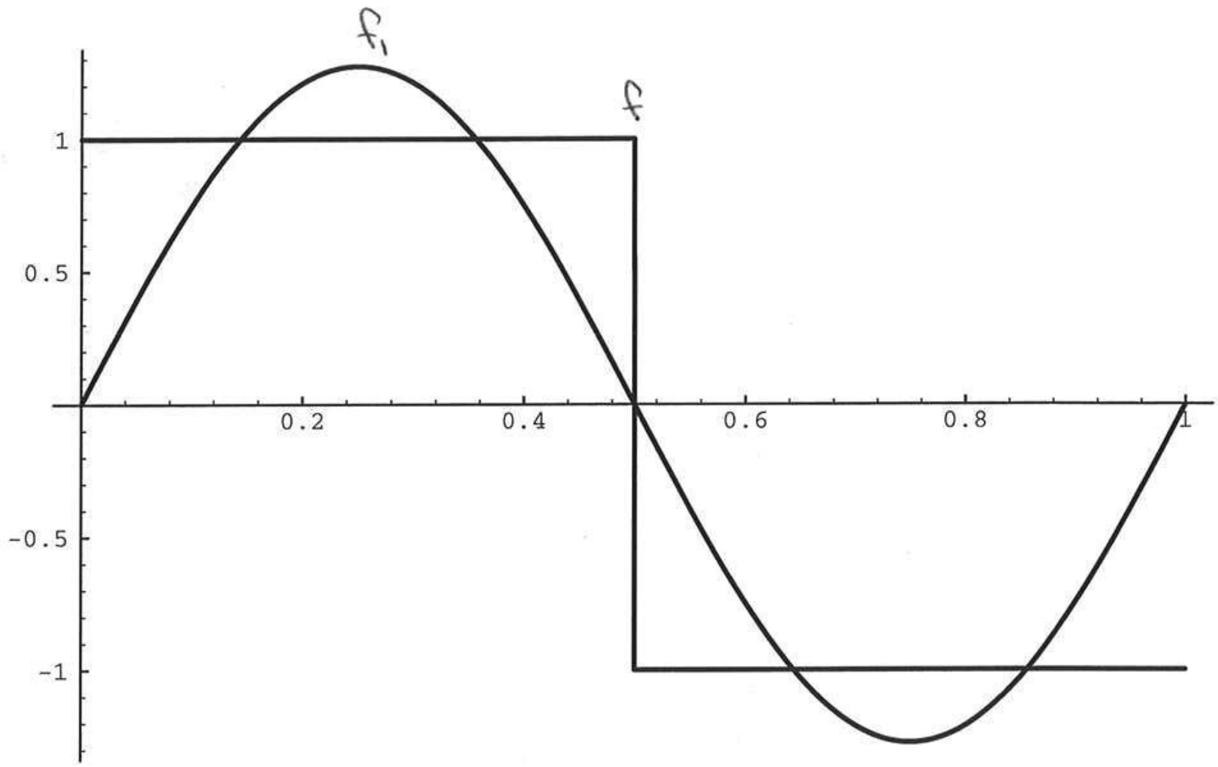
Then

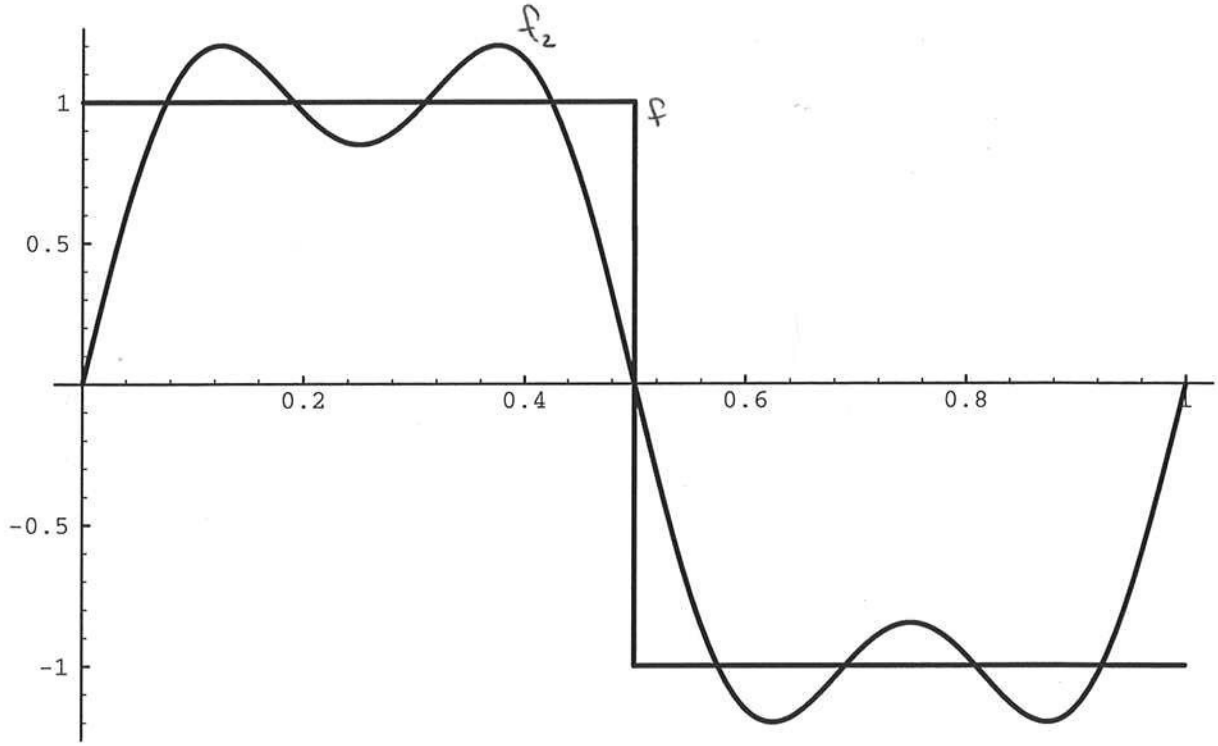
f_n converges to g pointwise

and in L^1-norm

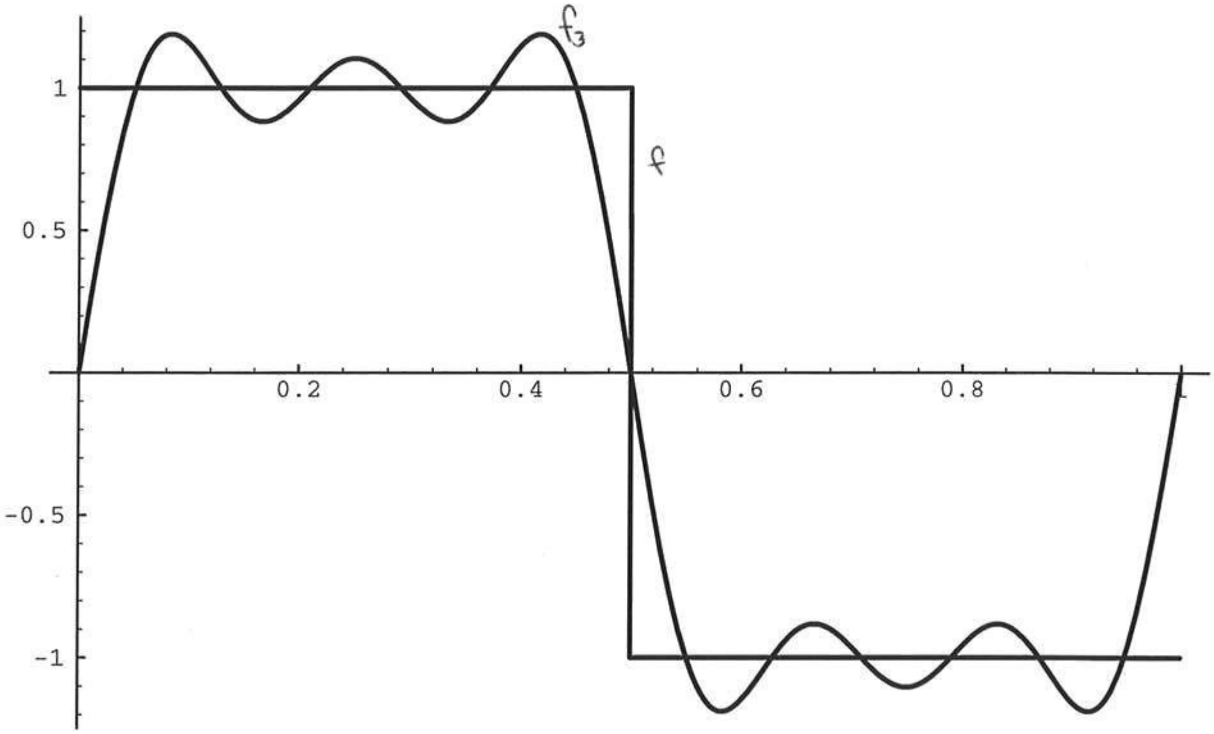
but not uniformly!

See figures showing $f_1, f_2, f_3, f_{10}, f_{50}$

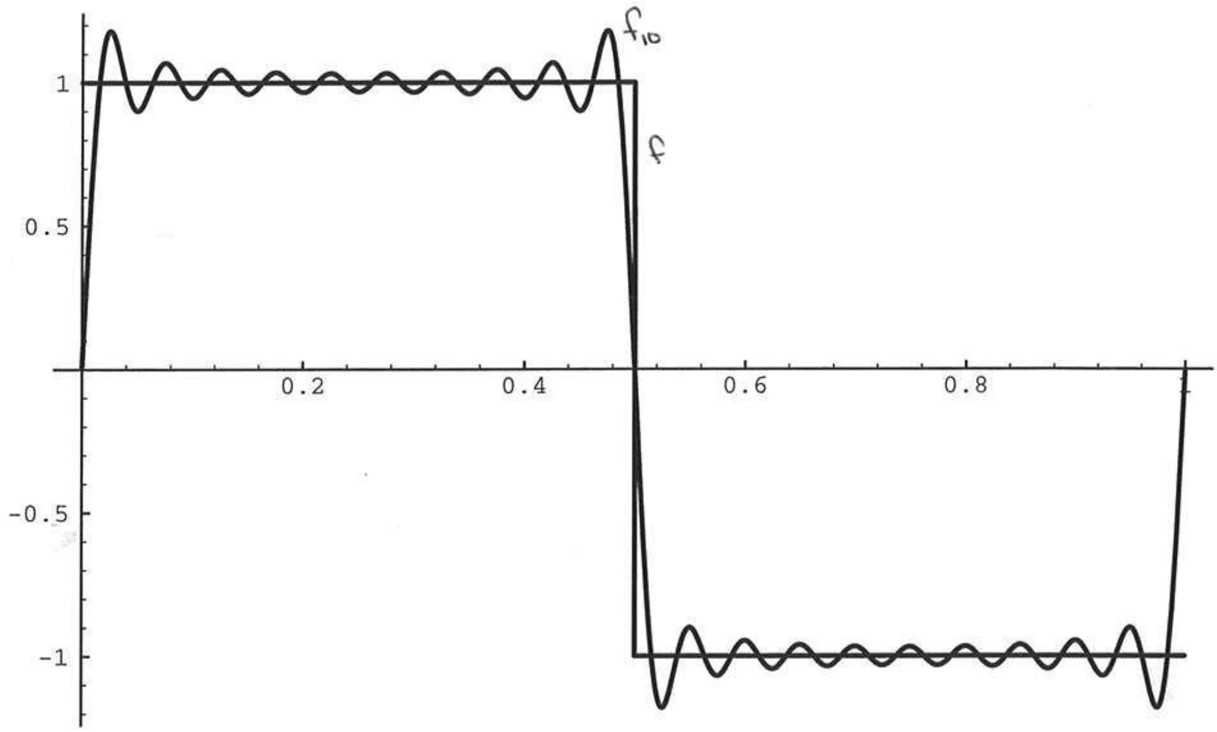

f_1
f
1
0.5
-0.5
-1
0.2
0.4
0.6
0.8
1



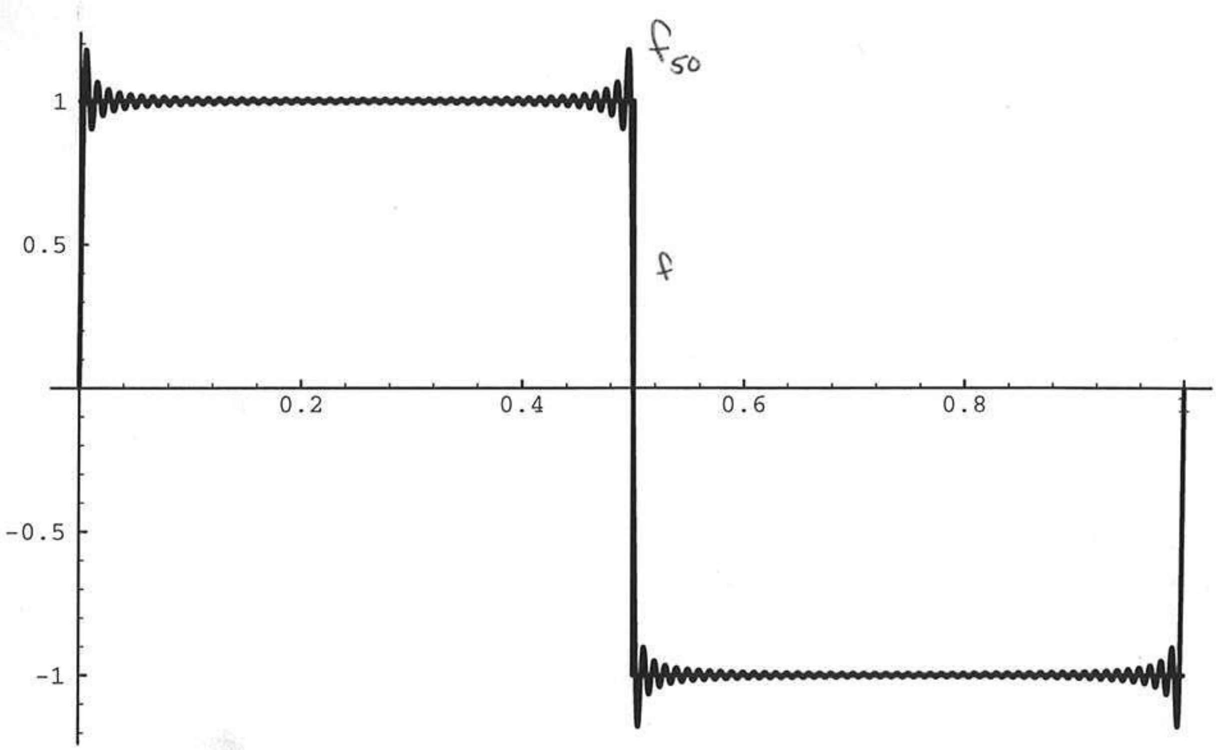
f_2
f
1
0.5
-0.5
-1
0.2
0.4
0.6
0.8
1



1
0.5
-0.5
-1
0.2
0.4
0.6
0.8
1
f_3
f



f_{10}
f
1
0.5
-0.5
-1
0.2
0.4
0.6
0.8
1



f_{50}
f
1
0.5
-0.5
-1
0.2
0.4
0.6
0.8
1

What does it mean for a sequence of functions (f_n) to converge to a limit function g?

Pointwise Convergence

For each individual x,

$$\lim_{n\to\infty} f_n(x) = g(x)$$

Book calls this "convergence"

Norm Convergence

All the x's are somehow involved

$$\lim_{n\to\infty} \|f_n - g\| = 0$$

L^1

L^2

L^∞

other norms

"uniform convergence"

Comparison

Pointwise convergence

$\forall x, \ f_n(x) \to f(x)$. Means: $\forall x, \ \forall \varepsilon > 0 \ \exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

N depends on x (& ε)

For a given ε, Each x many have a different N

Uniform Convergence

$f_n \to f$ uniformly means: $\|f_n - f\|_\infty \to 0$

$\sup_x |f_n(x) - f(x)| \to 0$

$\forall \varepsilon \ \exists N$ s.t. $n \geq N \Rightarrow \sup_x |f_n(x) - f(x)| = \|f_n - f\|_\infty < \varepsilon$

$\forall \varepsilon \ \exists N$ s.t. $\forall x, \ n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

Given ε, one N works for <u>all</u> x simultaneously
"uniform"

L^1 convergence

$f_n \to f$ in L^1 norm means $\|f_n - f\|_1 \to 0$

$\forall \varepsilon \ \exists N$ s.t. $n \geq N \Rightarrow \underbrace{\int |f_n(x) - f(x)| \, dx < \varepsilon.}_{\text{doesn't require } |f_n(x) - f(x)| < \varepsilon \ \forall x.}$

For L^1 or L^∞: Convergence $\Rightarrow$ Cauchy.
Same proof, true for any norm.

Problem: Does Cauchy $\Rightarrow$ Convergent? Check uniform norm case.

Yes. Proof:

sequence of functions.

Assume (f_n) is Cauchy in $L^\infty(\mathbb{R})$-norm (uniform norm).

Choose $\varepsilon > 0$. $\exists N > 0$ st. $n \geq N \Rightarrow \quad \|f_m - f_n\|_\infty < \varepsilon$.

or $\quad \sup_x |f_m(x) - f_n(x)| < \varepsilon$.

Hence $m, n \geq N \Rightarrow \quad |f_m(x) - f_n(x)| \leq \sup_x |f_m(x) - f_n(x)| < \varepsilon. \quad \forall x$.

Thus for each individual x, . $(f_n(x))$ is Cauchy.

sequence of numbers.

Define $f(x) = \lim_{n \to \infty} f_n(x)$.

We have then that $f_n \longrightarrow f$ Pointwise.

Claim: $f_n \longrightarrow f$ uniformly.

To see this, let m, x be fixed. Then we know (f_n) is Cauchy in L^∞, so

$a_n = \|f_m(x) - f_n(x)\|_\infty < \varepsilon \qquad \forall n \geq N$.

Hence $\lim\limits_{n\to\infty} |f_m(x) - f_n(x)| = \lim\limits_{n\to\infty} a_n \leq \varepsilon.$

$\uparrow$ each $a_n < \varepsilon$.

But $f_m(x)$ is <u>fixed</u>.

$$\underbrace{\lim_{n\to\infty} |f_m(x) - f_n(x)|}_{\leq \varepsilon} = \left|\lim_{n\to\infty} (f_m(x) - f_n(x))\right|$$

$$= \left| f_m(x) - \lim_{n\to\infty} f_n(x) \right|$$

$$= |f_m(x) - f(x)|$$

So $\uparrow \leq \varepsilon$.

Thus $\forall x$, $|f_m(x) - f(x)| \leq \varepsilon$.

So $\sup\limits_{x\in\mathbb{R}} |f_m(x) - f(x)| \leq \varepsilon$.

So $\|f_m - f\|_\infty \leq \varepsilon \quad \forall m \geq N$.

Thus $f_m \to f$ in L^∞ norm (uniformly).

~~Exercise: ℓ^2 is complete.~~