17. Convergence

Review: There are two basic types of convergence in $\mathbb{R}^p$:

(a) Norm Convergence

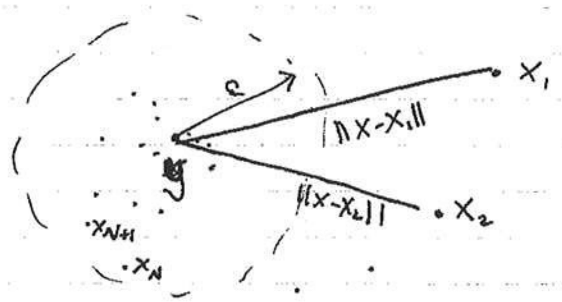
Let $(x_n) = (x_1, x_2, x_3, \ldots)$ be a sequence of vectors in $\mathbb{R}^p$.

Let $\|x\|$ be a given norm in $\mathbb{R}^p$.

Then (x_n) <u>converges in norm</u> to $y \in \mathbb{R}^p$ if $\lim\limits_{n\to\infty} \|y - x_n\| = 0$.

That is:

$$\forall \varepsilon > 0 \ \exists N > 0 \text{ s.t. } n \geq N \implies \|y - x_n\| < \varepsilon.$$



Also write: $x_n \longrightarrow y$ or $y = \lim\limits_{n\to\infty} x_n$.

<u>Remark:</u> All norms on the finite-dimensional space $\mathbb{R}^p$ are <u>equivalent</u>.

That is, if $\|x\|$ & $\|x\|'$ are two norms, then

$\exists\, A, B > 0$ s.t.

$$\forall x \in \mathbb{R}^p, \quad A\|x\| \leq \|x\|' \leq B\|x\|.$$

Exercise: Therefore $\lim \|y - x_n\| = 0 \iff \lim \|y - x_n\|' = 0$

(b) Componentwise Convergence

Again let $(x_n) = (x_1, x_2, \ldots)$ be a sequence of vectors in $\mathbb{R}^p$.

Write out the components explicitly:

$$x_1 = (x_{1,1}, x_{1,2}, \ldots, x_{1,p})$$
$$x_2 = (x_{2,1}, x_{2,2}, \ldots, x_{2,p})$$
$$\vdots$$
$$x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,p})$$
$$\vdots$$

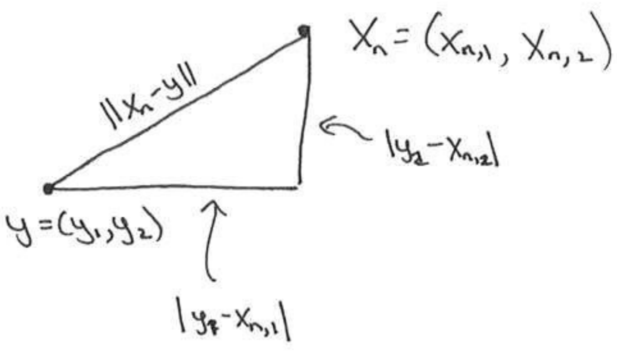
Write the components of y:

$$y = (y_1, y_2, \ldots, y_p).$$

Then (x_n) converges componentwise to y if

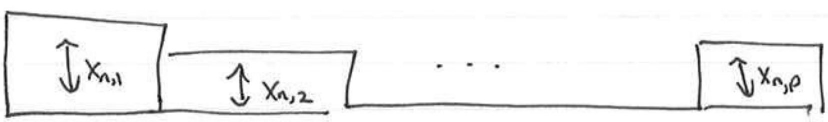
$$\begin{cases} \lim_{n\to\infty} x_{n,1} = y_1 & 1^{st} \text{ components of } x_n\text{'s converge to } 1^{st} \text{ comp of } y \\ \quad\vdots & \qquad\vdots \\ \lim_{n\to\infty} x_{n,p} = y_p & p^{th} \text{ '' '' '' '' } p^{th} \text{ '' '' } \end{cases}$$

Ex: $\mathbb{R}^2$



<u>Theorem:</u>
In $\mathbb{R}^p$: $x_n \to y$ in norm $\iff$ $x_n \to y$ componentwise.

Illustration as digital signals:



Each height converges individually $\iff$ Areas of all converge.

But this theorem does not generalize to ∞-dimensional spaces!

Convergence in ℓ^1

$$\ell^1 = \left\{ x = (x_1, x_2, \dots) \; : \; \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty \right\}$$

Consider a sequence of vectors in ℓ^1 (be careful to distinguish sequence index from component index).

Sequence $(x_n) = (x_1, x_2, \dots)$ in ℓ^1 – Now $x_n \in \ell^1$!

$$x_1 = (x_{1,1}, x_{1,2}, x_{1,3}, \dots) = (x_{1,k})_{k=1}^{\infty}$$

$$\vdots$$

$$x_n = (x_{n,1}, x_{n,2}, x_{n,3}, \dots) = (x_{n,k})_{k=1}^{\infty}$$

$$\vdots$$

Let

$$y = (y_1, y_2, y_3, \dots) = (y_k)_{k=1}^{\infty}$$

Then $x_n \to y$ <u>in norm</u> if $\displaystyle\lim_{n\to\infty} \|x_n - y\|_1 = 0$

or $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{\infty} |x_{n,k} - y_k| = 0$. (*)

And $x_n \to y$ <u>componentwise</u> if

$$\forall k, \quad \lim_{n\to\infty} x_{n,k} = y_k.$$

or

$$\forall k, \quad \lim_{n\to\infty} |x_{n,k} - y_k| = 0. \qquad (**)$$

Compare $(*)$ & $(**)$!

<u>Exercise:</u> $x_n \to y$ in norm $\Longrightarrow$ $x_n \to y$ componentwise.

<u>Example:</u> $\Longleftarrow$ is FALSE.

Set
$$\begin{aligned} x_1 &= (1, 0, 0, \dots) \\ x_2 &= (0, 1, 0, \dots) \\ x_3 &= (0, 0, 1, \dots) \\ &\vdots \\ y &= (0, 0, 0, \dots) \end{aligned}$$

Then $\forall k \; x_{n,k} \longrightarrow 0 = y_k$.

Converges componentwise!

But $\|x_n - y\|_1 = 1 \not\to 0$!

Does NOT converge in norm!

<u>Remark:</u> Similar for ℓ^2 or ℓ^∞. ℓ^∞ norm convergence is called <u>uniform convergence</u>.

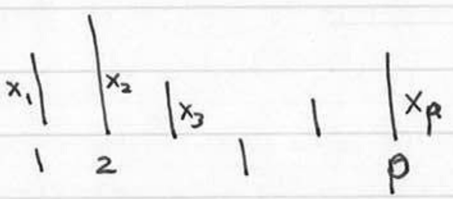
Convergence of Functions

Consider a sequence of functions $(f_n) = (f_1, f_2, \ldots)$ on some domain – for simplicity we'll take the domain to be $[0,1]$, but it could just as well be arbitrary. The functions will be real-valued, i.e., $f_n : [0,1] \longrightarrow \mathbb{R}$.

There are many different ways in which the sequence of functions (f_n) might converge to a limit function f. Unlike $\mathbb{R}^p$, where componentwise convergence & norm convergence are completely equivalent, for functions there many ways – all different & all reasonable – to define convergence. The analogue of componentwise convergence is <u>pointwise convergence</u>, and additionally there are (infinitely) many different types of norm convergence.

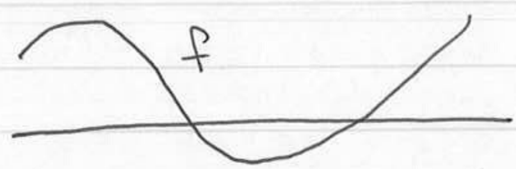
We'll illustrate this with some examples.

Recall: The components of a vector $x = (x_1, \dots, x_p)$ are analogous to function values $f(x)$ of f:



a vector $x = (x_1, \dots, x_p)$ is really a function $x : \{1, \dots, p\} \to \mathbb{R}$

vector = discrete function



a function $f : [0,1] \to \mathbb{R}$ has a continuous domain instead of a discrete domain

Componentwise convergence means convergence of the individual components:

$x_n = (x_{n,1}, \dots, x_{n,p})$ converges componentwise to $y = (y_1, \dots, y_p)$ if

$$\lim_{n \to \infty} x_{n,k} = y_k$$

for $k = 1, \dots, p$

Pointwise convergence means convergence of each function value:

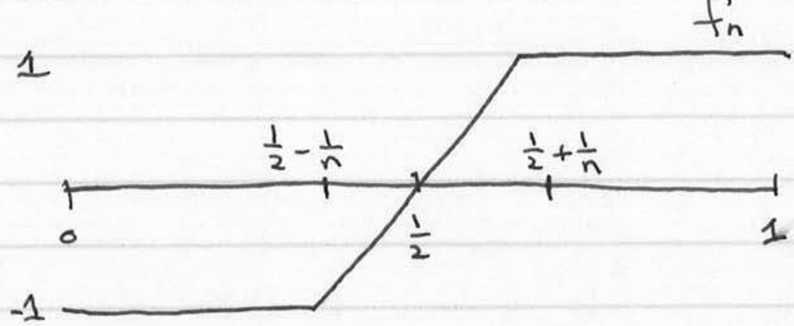
f_n converges pointwise to g if

$$\lim_{n \to \infty} f_n(x) = g(x)$$

for each $x \in [0,1]$.

Example

Let f_n be the continuous function on $[0,1]$ whose graph is:



Exercise: Write down an explicit formula for f_n:

$$f_n(x) = \begin{cases} 1, & \frac{1}{2}+\frac{1}{n} \leq x \leq 1 \\ ?, & \frac{1}{2}-\frac{1}{n} \leq x \leq \frac{1}{2}+\frac{1}{n} \\ -1, & 0 \leq x \leq \frac{1}{2}-\frac{1}{n} \end{cases}$$

If $\frac{1}{2} < x \leq 1$, then eventually n will be large enough that $\frac{1}{2}+\frac{1}{n} < x$. From some n onward, every $f_n(x)$ will equal 1:

$$\exists N \text{ s.t. } n > N \implies f_n(x) = 1$$

So for any particular x in the range $\frac{1}{2} < x \leq 1$,

$$\lim_{n\to\infty} f_n(x) = 1.$$

functions can be discontinuous!

Here is another example of poinwise convergence.

Example

Define f_n to be

$$f_n(x) = \begin{cases} ?, & 0 \leq x \leq \frac{1}{2n} \\ ?, & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1. \end{cases}$$

If we choose any $0 < x \leq 1$, then for all n large enough, $0 < \frac{1}{n} < x$, so

$$\exists N \text{ s.t. } n > N \Rightarrow f_n(x) = 0.$$

Thus for all $x > 0$ we have $\lim\limits_{n \to \infty} f_n(x) = 0$. And for $x = 0$ we have $f_n(0) = 0 \ \forall n$, so $\lim\limits_{n\to\infty} f_n(0) = 0$.

Therefore $\lim\limits_{n \to \infty} f_n(x) = 0 \quad \forall x \in [0,1]$, so

Likewise, show that

For $0 \le x < \frac{1}{2}$ we have $\lim_{n\to\infty} f_n(x) = -1$.

And for $x = \frac{1}{2}$, $f_n(\frac{1}{2}) = 0$ for every n, so

$$\lim_{n\to\infty} f_n(\tfrac{1}{2}) = 0.$$

Thus for each $x \in [0,1]$, $f_n(x)$ converges as $n \to \infty$, &

$$\lim_{n\to\infty} f_n(x) = g(x) = \begin{cases} 1, & \frac{1}{2} < x \le 1 \\ 0, & x = \frac{1}{2} \\ -1, & 0 \le x < \frac{1}{2} \end{cases}$$

$$\boxed{\text{For each } x, \quad f_n(x) \text{ converges to } g(x)}$$

This is pointwise convergence:

$$\boxed{f_n \text{ converges pointwise to } g}$$

Sometimes for emphasis we will write that

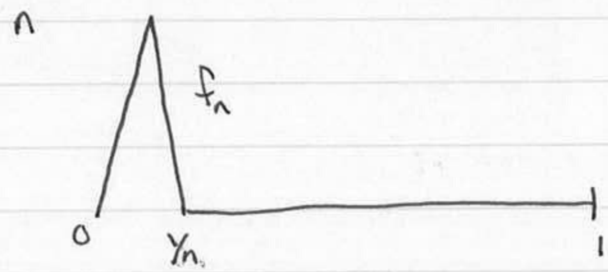
"$f_n(x)$ converges pointwise to $g(x)$ for each x."

Note that the pointwise limit of continuous

$\boxed{f_n \text{ converges pointwise to the zero function}}$

Exercise

Show that if f_n grows taller with n:

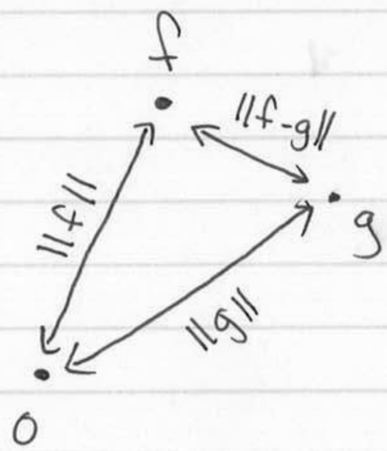


then we still have that f_n converges pointwise to zero.

Norm Convergence: $L^1, L^2,$ & uniform norms

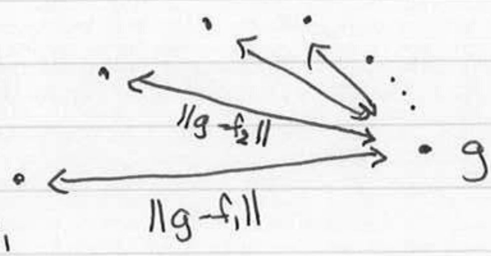
Pointwise convergence is just one type of convergence of functions. Another is norm convergence. But there isn't just one type of norm convergence — there are infinitely many different norms.

The idea of norm convergence is that we find a norm that somehow measures the "distance" between two functions f & g. We think of f & g as being two points in a vector space V, & find some way of measuring the lengths of vectors in V:



The distance between f & g is $\|f-g\|$, determined according to whatever the formula for that particular norm is.

Convergence with respect to the norm just means that the distance between f_n & g shrinks to zero as $n \to \infty$:



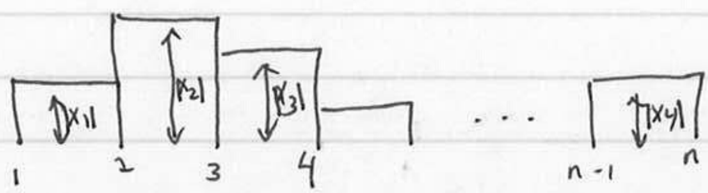
The distance between f_n & g is $\|g - f_n\|$, but its definition is given by the rule for the norm, it's not a physical distance as the Euclidean distance between points in $\mathbb{R}^n$ is.

The norms we most often use are not so strange – they're really just continuous versions of the l^p-norms on $\mathbb{R}^n$.

L^1-norm convergence

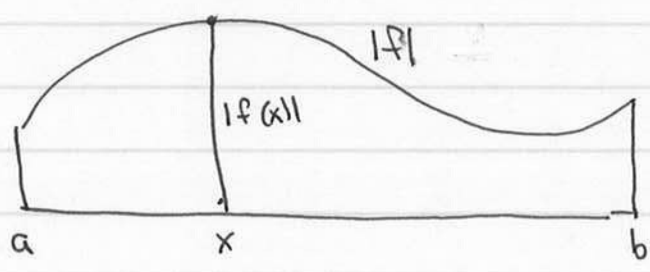
The L^1-norm for functions is a generalization of the ℓ^1-norm for vectors in $\mathbb{R}^n$, or the ℓ^1-norm for infinite sequences.

Contrast a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, which is a discrete function, with a function f on a domain $D \subseteq \mathbb{R}$ (e.g., $D = [0,1]$ or $\mathbb{R}$).



area of each box is $|x_k|$

total area is $\|x\|_1 = \sum_{k=1}^{n} |x_k| = |x_1| + \cdots + |x_n|$



total area is $\|f\|_1 = \int_a^b |f(x)|\, dx$

Definition

Let $D \subseteq \mathbb{R}$ be given. The L^1-norm of a function

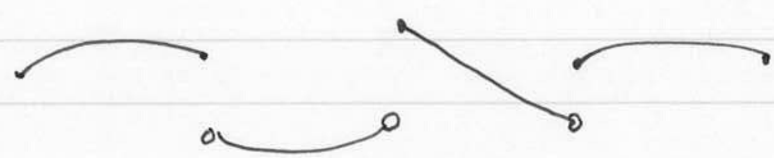
$f: D \rightarrow \mathbb{R}$ is

$$\|f\|_1 = \int_D |f(x)| \, dx.$$

Technical Problem

The meaning of the integral for completely arbitrary functions is subtle, & is covered in a course on measure theory & integration. In fact, there exist "nonmeasurable" functions that cannot be integrated.

To avoid complications, we shall only apply the integral to functions that are piecewise continuous:



a piecewise continuous function

In this case, the integral is just the ordinary Riemann integral we learned in Calculus.

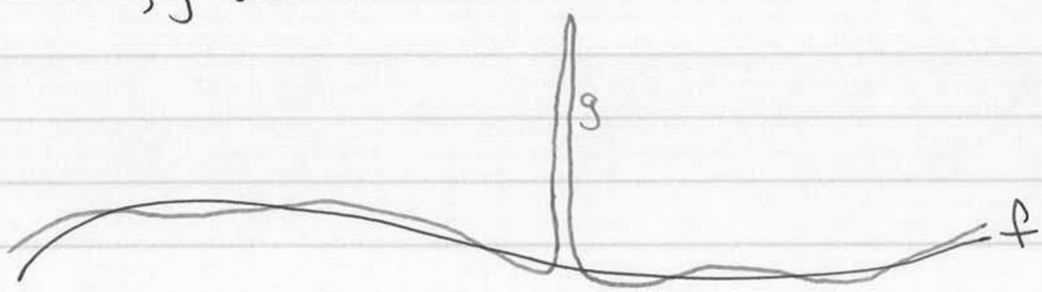
For more general functions, the Lebesgue Integral extends the Riemann integral to much more general functions.

The distance between two functions f & g, <u>measured using the L^1-norm</u>, is

$$\|f-g\|_1 = \int |f(x)-g(x)|\,dx$$

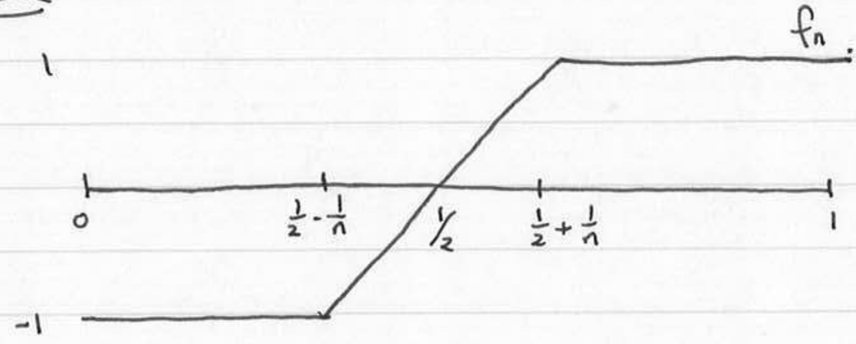
$$= \text{area } \underline{\text{between}} \text{ the graphs of } f \text{ \& } g.$$

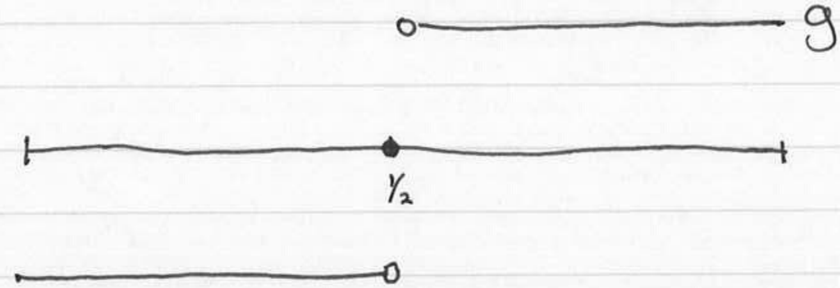
Example:
If f, g look like this:



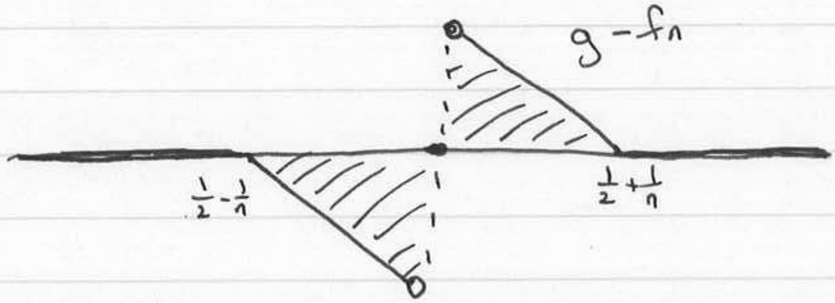
then the distance $\|f-g\|_1$ can be very small even though there's an x where f(x) & g(x) are far apart - it just depends on the total <u>area</u> between f & g.

Example





The function $g - f_n$ looks like this:



Therefore, $\|g - f_n\|_1 = \int_0^1 |g(x) - f_n(x)| \, dx$

is the area of the shaded region above. The distance between f_n & g is the sum of the areas of the two triangles:

$$\|g - f_n\|_1 = \int_0^1 |g(x) - f_n(x)|\,dx$$

$$= \tfrac{1}{2}\,\text{base} \times \text{height} + \tfrac{1}{2}\,\text{base} \times \text{height}$$

$$= \frac{1}{2} \cdot \frac{1}{n} \cdot 1 + \frac{1}{2} \cdot \frac{1}{n} \cdot 1$$

$$= \frac{1}{n}.$$

These two functions are a distance $\frac{1}{n}$ apart, according to the L^1-norm definition of distance.

Since the distance from f_n to g goes to zero:

$$\lim_{n\to\infty} \|g - f_n\|_1 = \lim_{n\to\infty} \frac{1}{n} = 0$$

we say that f_n converges to g in L^1-norm.

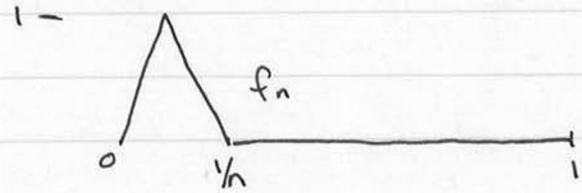
<u>Definition</u>

A sequence of piecewise continuous functions f_n <u>converges</u> to a piecewise continuous function g <u>in L^1-norm</u> if

$$\lim_{n\to\infty} \|g - f_n\|_1 = 0.$$

We write $f_n \to g$ in L^1-norm.

Example



If f_n is as above & $g = 0$, then

$$\|g - f_n\|_1 = \int_0^1 |g(x) - f_n(x)|\, dx$$

$$= \int_0^1 |f_n(x)|\, dx$$

$$= \frac{1}{2} \cdot \frac{1}{n} \cdot 1$$

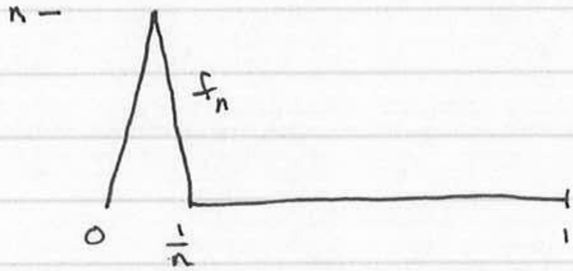
$$= \frac{1}{2n}$$

$$\longrightarrow 0 \text{ as } n \to \infty,$$

so $f_n \longrightarrow 0$ in L^1-norm.

Remember that an earlier example showed that $f_n \to 0$ pointwise, but these are two different types of convergence! Compare with the next example.

Example



A previous example showed that if f_n is as above, then $f_n \longrightarrow 0$ pointwise. However, the L^1-distance from f_n to 0 is

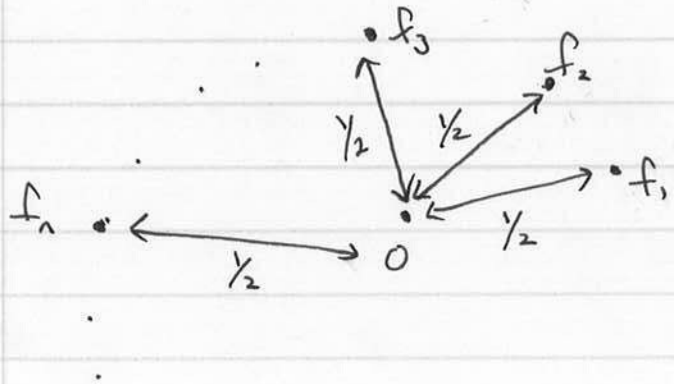
$$\|0 - f_n\|_1 = \int_0^1 |0 - f_n(x)|\, dx$$

$$= \int_0^1 |f_n(x)|\, dx$$

$$= \frac{1}{2} \cdot \frac{1}{n} \cdot n$$

$$= \frac{1}{2} \quad \text{for } \underline{\text{all}}\ n.$$

Each f_n is $\frac{1}{2}$ unit away from 0, according to the L^1-norm!



So f_n never gets close to 0, according to the L^1-norm! $f_n \not\to 0$ in L^1-norm, even though f_n converges pointwise to 0.

In general:

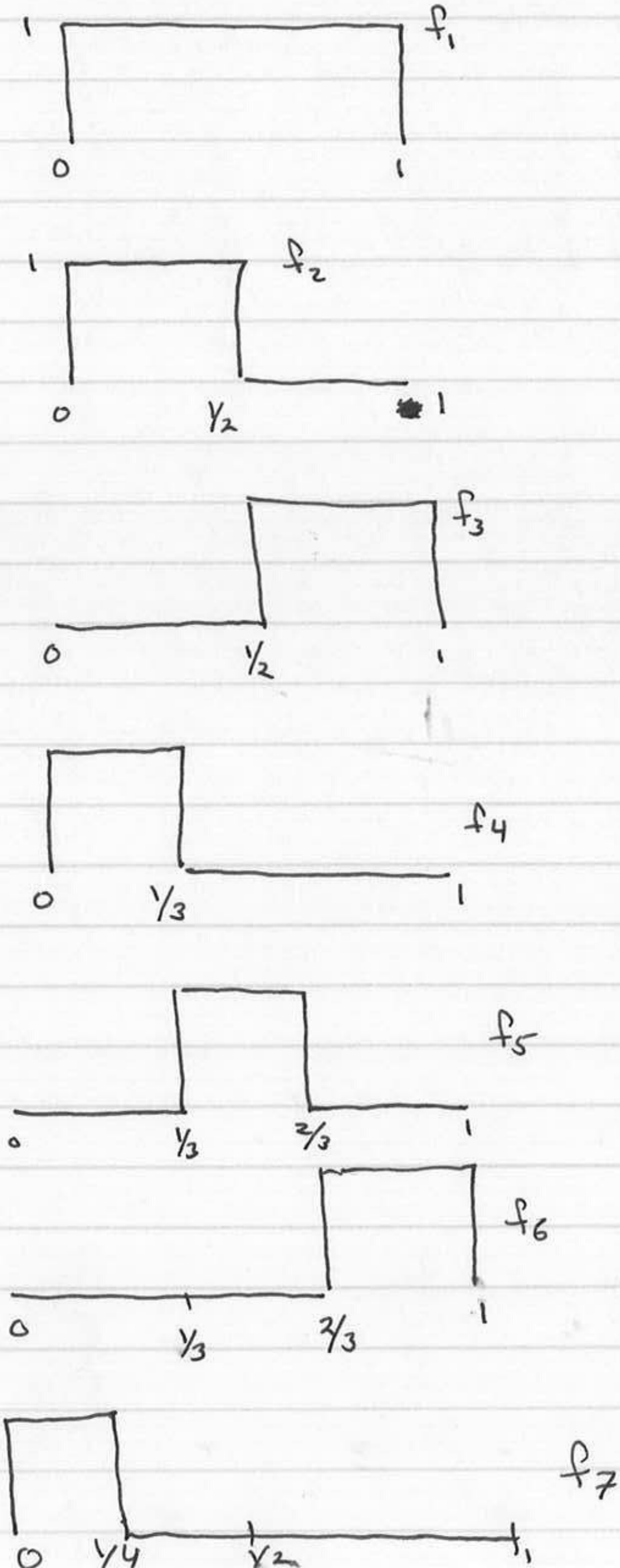
$$f_n \to f \text{ in } L^1\text{-norm} \;\underset{\not\Leftarrow}{\not\Rightarrow}\; f_n \to f \text{ pointwise}$$

Neither type of convergence implies the other!

We've seen an example (the preceding one) of functions f_n s.t. $f_n \to 0$ pointwise but $f_n \not\to 0$ in L^1-norm. Now we give an example of functions f_n s.t. $f_n \to 0$ in L^1-norm but $f_n \not\to 0$ pointwise.

<u>Example</u>: The "Marching Boxes" example.

Consider $f_1, f_2, f_3, \ldots$ like this:



etc.

Then $f_n \to 0$ in L^1-norm, because

$$\|0 - f_1\|_1 = \|f_1\|_1 = \int_0^1 |f_1(x)|\,dx = 1$$

$$\|0 - f_2\|_1 = \|f_2\|_1 = \frac{1}{2}$$

$$\|0 - f_3\|_1 = \|f_3\|_1 = \frac{1}{2}$$

$$\|0 - f_4\|_1 = \|f_4\|_1 = \frac{1}{3}$$

$$\|0 - f_5\|_1 = \|f_5\|_1 = \frac{1}{3}$$

$$\|0 - f_6\|_1 = \|f_6\|_1 = \frac{1}{3}$$

$$\frac{1}{4}$$

$$\frac{1}{4}$$

$$\frac{1}{4}$$

$$\frac{1}{4}$$

etc.

$\|0 - f_n\|_1 = \|f_n\|_1 \longrightarrow 0$ as $n \to \infty$,

so $f_n \to 0$ in L^1-norm.

But f_n does NOT converge to 0 (or

anything else) pointwise!! Why??

L^2-norm

The L^2-norm for functions generalizes the Euclidean norm for vectors in $\mathbb{R}^n$. The L^2-norm of a piecewise continuous function f on a domain D is

$$\|f\|_2 = \left(\int_D |f(x)|^2 \, dx \right)^{1/2}.$$

We say that $f_n \to g$ <u>in L^2-norm</u> if

$$\lim_{n\to\infty} \|g - f_n\|_2 = \lim_{n\to\infty} \left(\int_D |g(x) - f_n(x)|^2 \, dx \right)^{1/2}$$

$$= 0$$

Exercise: Recheck the preceding examples to see if we have L^2-norm convergence.

Note

On a finite domain, such as $[0,1]$, there is a relation between the L^1 & L^2 norms. Suppose $f:[0,1] \to \mathbb{R}$ is piecewise continuous. Then

$$\|f\|_1 = \int_0^1 |f(x)|\,dx$$

$$= \int_0^1 |f(x)| \cdot 1\,dx$$

$$= \langle |f|, 1 \rangle$$ inner product of $|f|$ & the constant function 1

$$\leq \| |f| \|_2 \, \|1\|_2$$ Cauchy-Schwarz inequality

$$= \left(\int_0^1 |f(x)|^2 dx\right)^{1/2} \left(\int_0^1 1^2\,dx\right)^{1/2}$$

$$= \|f\|_2 \cdot 1$$

$$= \|f\|_2 .$$

In particular,

$$\|g - f_n\|_1 \leq \|g - f_n\|_2$$

for functions on the domain $[0,1]$.