15. Subsequences & Combinations

Given $X = (x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \dots)$

Given indices $n_1 < n_2 < n_3 < \cdots$

Let $y_k = x_{n_k}, \quad k = 1, 2, \dots$

Then
$$(y_k)_{k=1}^{\infty} = (y_1, y_2, y_3, \dots)$$
$$= (x_{n_1}, x_{n_2}, x_{n_3}, \dots)$$
$$= (x_{n_k})_{k=1}^{\infty}$$

is a subsequence of $(x_n)_{n=1}^{\infty}$.

<u>Ex:</u> $(\underset{y_1}{x_2}, \underset{y_2}{x_3}, \underset{y_3}{x_5}, x_7, x_{11}, \dots)$ is a subsequence

$(x_7, x_{11}, x_2, x_3, \dots)$ is <u>not</u> a subsequence.

Lemma

$x_n \to x \iff x_{n_k} \to x \ \forall$ subsequence (x_{n_k})

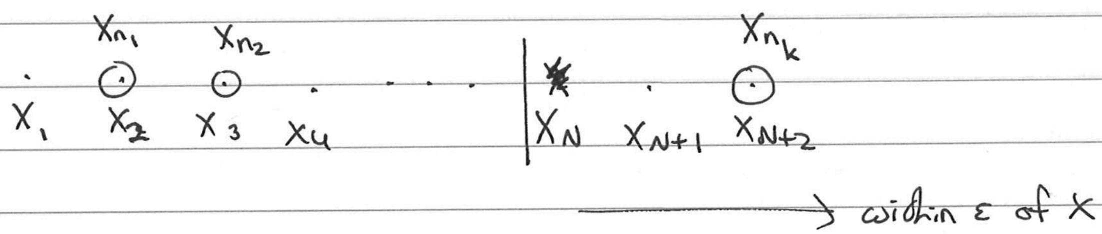
Proof:

$\Rightarrow$ Assume $x_n \to x$. Choose $\varepsilon > 0$. Must show:

Goal. $\boxed{\exists K > 0 \text{ s.t. } k > K \Rightarrow \|x - x_{n_k}\| < \varepsilon.}$

We know

$$\exists N > 0 \text{ st. } n > N \Rightarrow \|x - x_n\| < \varepsilon.$$



Since $n_1 < n_2 < \cdots$ is an increasing sequence of integers, $\exists K$ st. $n_K > N$.

Hence if $k > K$ then $n_k > n_K > N$, so

$$\|x - x_{n_k}\| < \varepsilon.$$

$\Leftarrow$ Trivial $n_k = k$: the full sequence is one of the subsequences.

Exercise:

Let (x_n) be a sequence in $\mathbb{R}^p$ and $x \in \mathbb{R}^p$.
Suppose that

$\forall$ subsequence (y_k) of (x_n),

$\exists$ subsequence (z_j) of (y_k) s.t. $z_j \longrightarrow x$.

Prove that $x_n \longrightarrow x$.

Hint: What does $x_n \not\to x$ mean? Proof by contradiction.