

14. Sequences

A sequence (with values in a set) S is a function from \mathbb{N} to S .

$$X: \mathbb{N} \rightarrow S \quad X = (x_1, x_2, \dots) = (x_n)$$
$$n \mapsto X(n) = x_n$$

We will deal with sequences whose values are in \mathbb{R}^p .
MOST RESULTS HOLD FOR ANY SPACE THAT HAS A NORM.

Sum of sequences $X = (x_n)$ & $Y = (y_n)$ is $X + Y = (x_n + y_n)$

Componentwise addition

Scalar Mult. $cX = (cx_n)$

$V = \{ \text{all sequences } X = (x_n) : x_n \in \mathbb{R}^p \}$ is a vector space

V is more than a vector space

Products: $XY = (x_n y_n)$ V is an algebra w.r.t. componentwise multiplication.

Quotients: $X/Y = (x_n/y_n)$ But not always defined!

"Inner Product" $X \cdot Y = (x_n \cdot y_n)$
(Bartle notation)

Beware:

INCONSISTENT NOTATION

This is not an inner product on V ! $X \cdot Y$ is a sequence, not a real number!

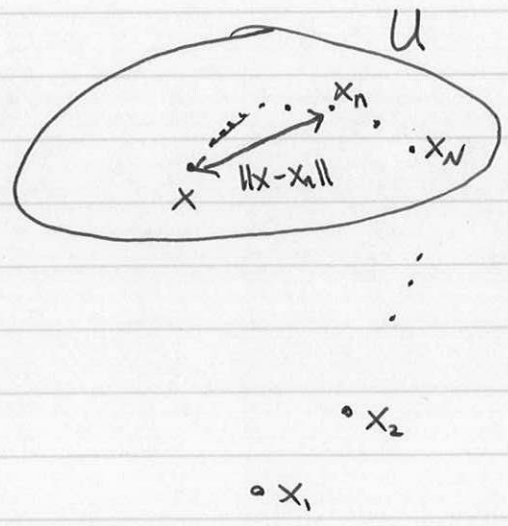
Definition: Limit of an \mathbb{R}^p -valued sequence (valid for any normed space)

Let $X = (x_n)$ be a sequence in \mathbb{R}^p .

Then $x \in \mathbb{R}^p$ is a limit of X if

\forall neighborhood U of x , $\exists N > 0$ s.t.

$$n \geq N \implies x_n \in U.$$



From some point onwards, you have to stay in U .

As usual when dealing with neighborhoods, every thing comes down in the end to open balls.

Exercise

Given $X = (x_n)$ & $x \in \mathbb{R}^p$, show TFAE.

- x is a limit of $X = (x_n)$.
- $\forall \varepsilon > 0 \exists N > 0$ st. $n \geq N \Rightarrow \|x - x_n\| < \varepsilon$.
- $\forall k \in \mathbb{N} \exists N > 0$ st. $n \geq N \Rightarrow \|x - x_n\| < \frac{1}{k}$.

Notation

We write any of

$$x_n \rightarrow x$$

$$\text{or } x = \lim_{n \rightarrow \infty} x_n = \lim_n x_n = \lim x_n = \lim X$$

to mean that x is a limit of $X = (x_n)$.

Exercise

- $X = \left(\frac{1}{n}\right)$ is convergent (has a limit)
- $X = ((-1)^n)$ is not convergent (has no limit).

Next we give another "Highlander Theorem."

Lemma (Valid for any normed space).

A sequence $X = (x_n)$ in \mathbb{R}^p can have at most one limit.

Proof:

Suppose $x_n \rightarrow x$ & $x_n \rightarrow y$. Choose any $\varepsilon > 0$.

Then:

$$\exists N_1 \text{ s.t. } n \geq N_1 \Rightarrow \|x - x_n\| < \varepsilon.$$

$$\exists N_2 \text{ s.t. } n \geq N_2 \Rightarrow \|y - x_n\| < \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

we have both $n \geq N_1$ & $n \geq N_2$, so

$$\begin{aligned} \|x - y\| &= \|x - x_n + x_n - y\| \\ &\leq \|x - x_n\| + \|x_n - y\| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon. \end{aligned}$$

Thus $0 \leq \|x - y\| < 2\varepsilon$ for every $\varepsilon > 0$, so

$\|x - y\| = 0$ & therefore $x = y$. \square

Definition

$X = (x_n)$ is bounded if $\exists M > 0$ s.t. $\|x_n\| \leq M \quad \forall n \in \mathbb{N}$

Exercise

$X = (x_n)$ is bounded if & only if $\sup_n \|x_n\| < \infty$.

Lemma (Valid for any normed space).

All convergent sequences are bounded.

Proof:

Suppose $x_n \rightarrow x$. Choose $\varepsilon = 1$. Then $\exists N$ s.t.

$$n \geq N \Rightarrow \|x - x_n\| < 1.$$

Hence $x_n \in B_1(x)$ for $n \geq N$, & for these n ,

$$\begin{aligned} \|x_n\| &= \|x_n - x + x\| \leq \|x_n - x\| + \|x\| \\ &< 1 + \|x\|. \end{aligned}$$

This is true for $n \geq N$ only. Therefore if n is any positive integer $\forall n$

$$\|x_n\| \leq \max \{ \|x_1\|, \dots, \|x_N\|, \|x\| + 1 \} \quad (\text{why?})$$

So X is bounded. \square

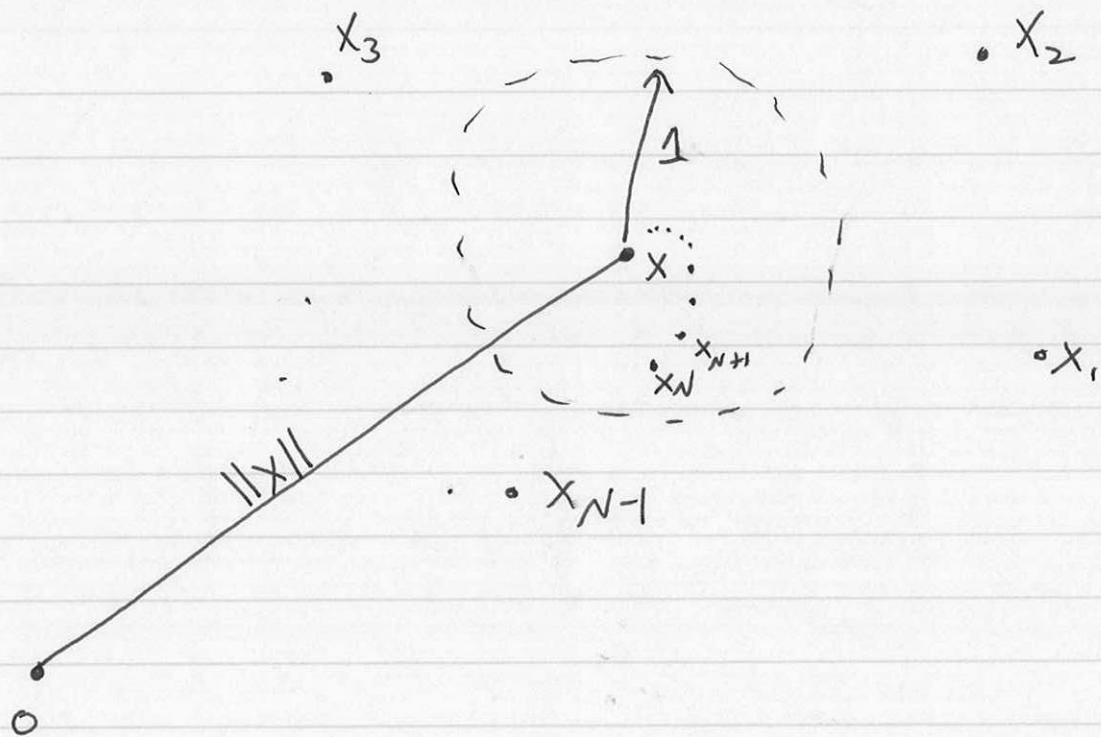


Illustration for proof of lemma.

Exercise: Not all bounded sequences converge!

Definition

Given a sequence $X = (x_n)$ of points in \mathbb{R}^p , write the components of x_n as

$$x_n = (x_{n,1}, x_{n,2}, \dots, x_{n,p})$$

Given $y = (y_1, \dots, y_p) \in \mathbb{R}^p$, we say that

x_n converges componentwise to y if

$$\lim_{n \rightarrow \infty} x_{n,k} = y_k \quad \text{for } k = 1, \dots, p.$$

Theorem (\Rightarrow holds more generally, \Leftarrow is because \mathbb{R}^p is finite-dimensional)

Given a sequence $X = (x_n)$ in \mathbb{R}^p & given $y \in \mathbb{R}^p$,

$$\begin{array}{ccc} x_n \rightarrow y & \iff & x_n \text{ Converges componentwise} \\ \text{(convergence of vectors)} & & \text{to } y \end{array}$$

Idea of proof:

\Rightarrow For each k ,

$$|x_{n,k} - y_k| \leq \|x_n - y\|$$

If $x_n \rightarrow y$ then RHS $\rightarrow 0$.

$$\Leftarrow \|x_n - y\| = \sqrt{(x_{n,1} - y_1)^2 + \dots + (x_{n,p} - y_p)^2}$$

Componentwise convergence implies LHS & RHS $\rightarrow 0$

(important: there are only finite many components!)

Exercise: Make this proof sketch precise.

Some Facts about sequences

$$\lim_{n \rightarrow \infty} x_n = x \quad [\text{limit of vectors } x_n \text{ in } \mathbb{R}^n]$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists N > 0 \text{ st. } n \geq N \Rightarrow \|x - x_n\| < \varepsilon$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists N > 0 \text{ st. } n \geq N \Rightarrow |\|x - x_n\| - 0| < \varepsilon$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \|x - x_n\| = 0 \quad [\text{limit of numbers } \|x - x_n\| \text{ is zero}]$$

Remark: $n > N$ or $n \geq N$: doesn't matter
 $\|x - x_n\| < \varepsilon$ or $\|x - x_n\| \leq \varepsilon$ doesn't matter } exercise

Lemma: Convergence of sums

$$x_n \rightarrow x \quad \& \quad y_n \rightarrow y \quad \Rightarrow \quad x_n + y_n \rightarrow x + y$$

Same with different notation:

$$\lim_{n \rightarrow \infty} x_n = x, \quad \lim_{n \rightarrow \infty} y_n = y \quad \Rightarrow \quad \lim_{n \rightarrow \infty} (x_n + y_n) = x + y$$

Exercise: Converse is false! Give a counterexample!

Proof of Lemma. Suppose $x_n \rightarrow x$ & $y_n \rightarrow y$.

Choose $\varepsilon > 0$. Then $\varepsilon/2 > 0$, so


$$\exists N_1 \text{ st. } n > N_1 \Rightarrow \|x - x_n\| < \varepsilon/2$$

$$\exists N_2 \text{ st. } n > N_2 \Rightarrow \|y - y_n\| < \varepsilon/2$$

Let $N = \max\{N_1, N_2\}$. If $n > N$ then we have both

$n > N_1$ & $n > N_2$, so

$$\begin{aligned} \|(x+y) - (x_n+y_n)\| &\leq \|x-x_n\| + \|y-y_n\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus $x_n + y_n \rightarrow x + y$. 

Lemma Convergence of products.

$$x_n \rightarrow x, \quad y_n \rightarrow y \quad \Rightarrow \quad x_n y_n \rightarrow xy$$

Note: These are componentwise products:

$$x_n y_n = (x_{n,1} y_{n,1}, \dots, x_{n,p} y_{n,p}).$$

Proof sketch for $p=1$

When $p=1$, x_n, y_n, x, y are all numbers, so

$x_n \rightarrow x, y_n \rightarrow y$ means that $|x-x_n| \rightarrow 0, |y-y_n| \rightarrow 0$

(absolute value). Then

$$\begin{aligned}
 |xy - x_n y_n| &= |xy - x_n y + x_n y - x_n y_n| \\
 &\leq |xy - x_n y| + |x_n y - x_n y_n| \\
 &\leq \underbrace{|y|}_{\text{constant}} \underbrace{|x - x_n|}_{\text{small for } n \text{ large}} + \underbrace{|x_n|}_{?} \underbrace{|y - y_n|}_{\text{small for } n \text{ large}} \quad (*)
 \end{aligned}$$

Choose ε . $\exists N_1$ s.t.

$$n > N_1 \implies |x - x_n| < \frac{\varepsilon}{2|y|} \quad (\text{What if } y=0??)$$

So the first term in (*) will be $< \frac{\varepsilon}{2}$ for n large.

For the second term, recall that convergence sequences

are bounded. So $\exists M$ s.t.

$$|x_n| \leq M \quad \forall n.$$

Then $\exists N_2$ s.t.

$$n > N_2 \implies |y - y_n| < \frac{\varepsilon}{2M}$$

So the 2nd term in (*) will be $< \frac{\varepsilon}{2}$ for n large.

For $n > N = \max \{N_1, N_2\}$ we have

$$\begin{aligned}
 (*) & \ll |y| \frac{\epsilon}{2|y|} + |x_n| \frac{\epsilon}{2M} \\
 & \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 & = \epsilon.
 \end{aligned}$$

Hence $x_n y_n \rightarrow xy$.

Exercise: Fix \mathbb{R} problem about $y=0$ & extend to general p . Hint: $x_n \rightarrow x$ if & only if x_n converges componentwise to x .

Exercise: Show \mathbb{R} converse is false, i.e., give an example of x_n, y_n, x, y s.t. $x_n y_n \rightarrow xy$ but $x_n \not\rightarrow x, y_n \not\rightarrow y$.