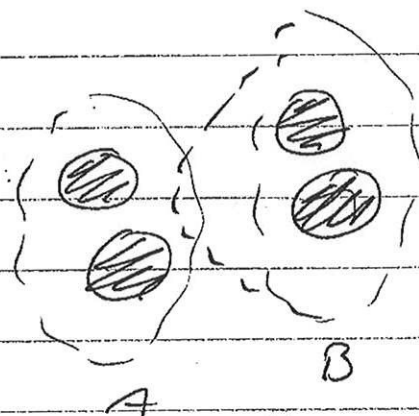


12 Connected Sets

Definition

$D \subseteq \mathbb{R}^p$ is disconnected if



\exists ^{nonempty} open A, B st.

$$A \cap D \neq \emptyset, B \cap D \neq \emptyset$$

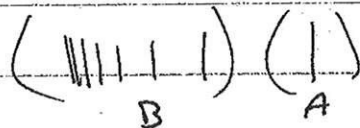
$$(A \cap D) \cap (B \cap D) = \emptyset$$

$$(A \cap D) \cup (B \cap D) = D$$

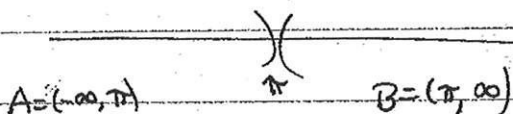
Then (A, B) is a disconnection of D .

$C \subseteq \mathbb{R}^p$ is connected if it is not disconnected.

Ex: $\{\frac{1}{n} : n \in \mathbb{N}\}$ is disconnected.



Ex: \mathbb{Q} is disconnected



Ex: $(0, 1)$ is connected

$[0, 1)$

$(0, 1]$

$[0, 1]$

Exercise: Connected, x cluster-pt $\Rightarrow C \cup \{x\}$ connec

Suppose (A, B) was a disconnection of $C \cup \{x\}$.

Show it would then be a disconnection of C .

Theorem: Any interval ^{in \mathbb{R}} is connected. 2

Proof: Suppose $I = (0, 1)$ is disconnected, let (A, B) be a disconnection.

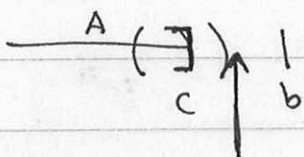
$I \cap A, I \cap B$ are not single points (they are open).

Let $a \in A, b \in B$. Either $a < b$ or $b < a$.

Assume $0 < a < b < 1$. Let $c = \sup \{x \in A : x < b\}$.

Note $a \leq c \leq b$ so $c \in I = A \cup B$.

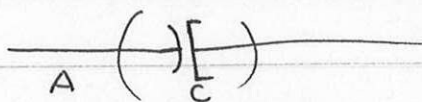
If $c \in A$ then $c \neq b$.



A is open! $(c - \epsilon, c + \epsilon) \subseteq A$

Contradiction.

If $c \in B$ then:



$(c - \epsilon, c + \epsilon) \subseteq B$

B is open!

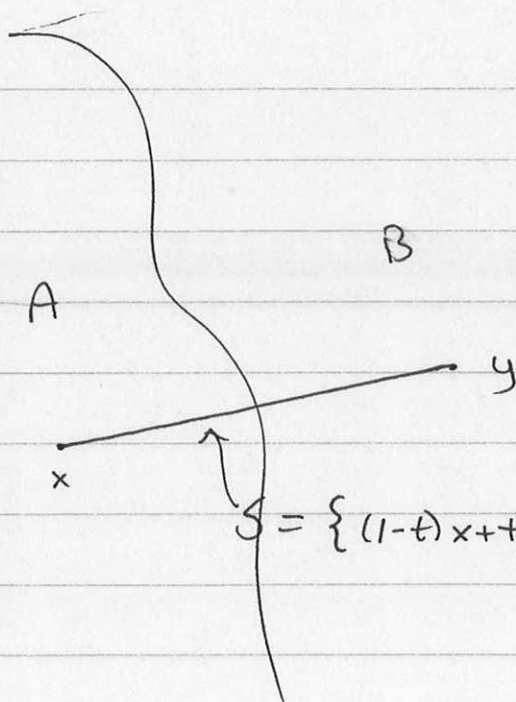
Contradiction. \square

Ex: \mathbb{R}^p is connected.

Proof:

Suppose not

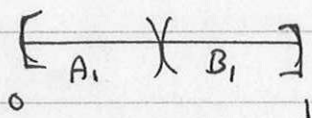
Choose $x \in A, y \in B$.



Let

$$A_1 = \{t : (1-t)x + ty \in A\}$$

$$B_1 = \{t : (1-t)x + ty \in B\}$$



Exercise: (A_1, B_1) is a disconnection of $[0, 1]$.

\square

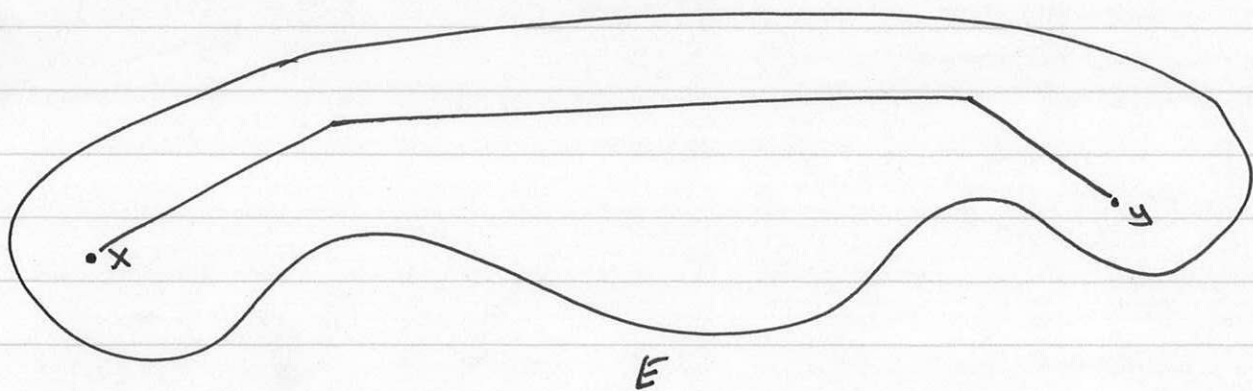
Corollary Only \emptyset, \mathbb{R}^p can be both open & closed.

Proof:

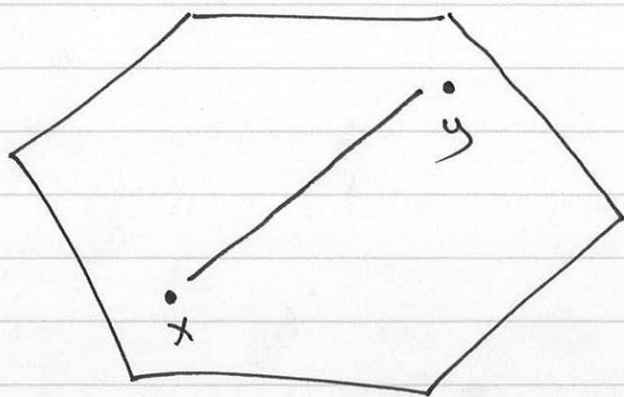
Suppose A is both open & closed. Then $e(A)$ is both open & closed. If $A \neq \emptyset$ then $(A, e(A))$ forms a disconnection of \mathbb{R}^p , which is impossible. \square

Definition

A set $E \subseteq \mathbb{R}^D$ is polygonally path-connected if every $x, y \in E$ can be joined by a polygonal curve lying entirely within E .



Example: All convex sets are polygonally path-connected.



convex means any two $x, y \in E$ can be joined by a line segment

Q. Are all connected sets polygonally path-connected?

Are all polygonally path-connected sets connected?

(5)

Example: $E = \{(x, x^2) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$

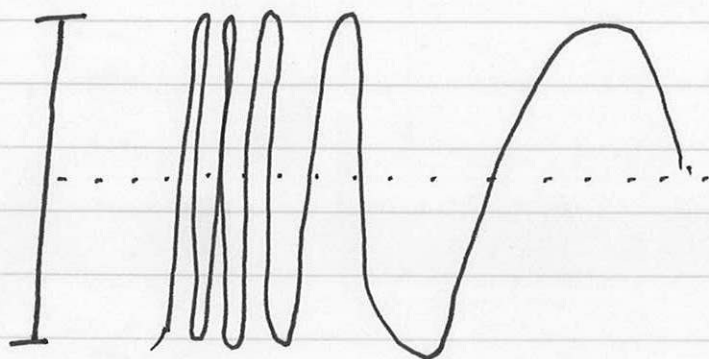


\mathbb{R}^2

E is connected by not polygonally path-connected.

Perhaps it be better to use any "continuous" curves to define path-connected. But consider:

Example: $E = \{(0, y) : -1 \leq y \leq 1\} \cup \{(x, \sin \frac{1}{x}) : x > 0\}$



oscillates faster & faster

Exercise: E is connected!

But clearly E is not path-connected, even using any continuous curves.

(6)

Theorem

If $G \subseteq \mathbb{R}^n$ is open then

G is connected $\iff G$ is polygonally path-connected.

Proof:

\Rightarrow Assume G is connected & choose any $x \in G$.

Set

$$G_1 = \left\{ y \in G : y \text{ can be joined to } x \text{ by a polygonal path contained in } G \right\}$$

$$G_2 = \left\{ z \in G : z \text{ cannot be joined to } x \text{ by a polygonal path contained in } G \right\}$$

We must show that $G_2 = \emptyset$.

$$\text{Note that } G_1 \cup G_2 = G \quad \& \quad G_1 \cap G_2 = \emptyset.$$

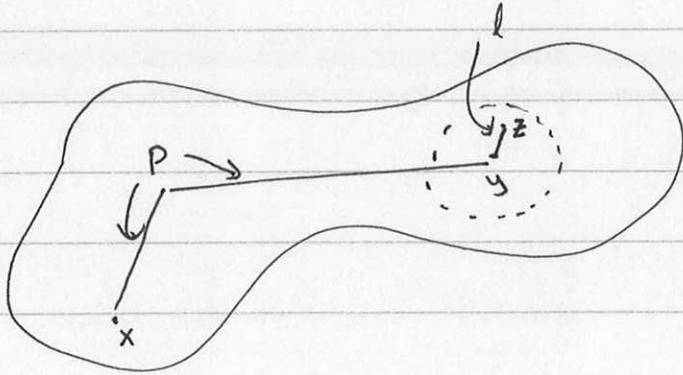
Claim: G_1 is open.

To see this, choose $y \in G_1$. Since G is open,

$\exists r > 0$ s.t. $B_r(y) \subseteq G$. We will show

$B_r(y) \subseteq G_1$. Choose $z \in B_r(y)$. We must show $z \in G_1$.

Let P be the polygonal curve ~~joining~~ joining x to y .



Note y is joined to z by a line segment^l lying in G .

Hence $P \cup l$ is a polygonal curve joining x to z .

Hence $z \in G_1$. So G_1 is open.

Claim: G_2 is open. Exercise.

Thus (G_1, G_2) is a disconnection of G if $G_1, G_2 \neq \emptyset$.

Hence one of G_1, G_2 must be empty. Since $x \in G_1$, it

can't be empty. Therefore $G_2 = \emptyset$.

~~← We'll prove the contrapositive: IF G is not connected then $\exists x, y \in G$ that cannot be joined by a polygonal curve lying in G .~~

~~Assume G is not connected. Then \exists disconnection (A, B) .~~

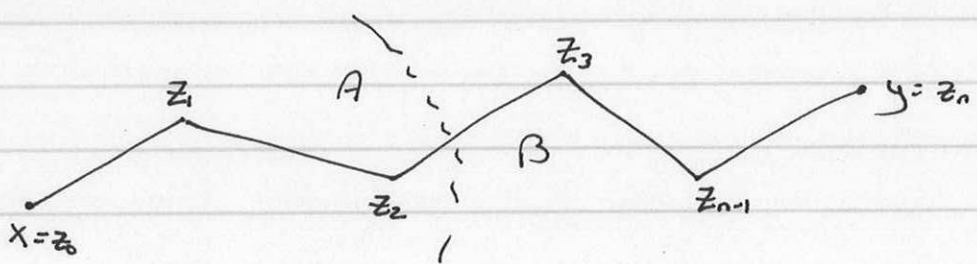
~~Let $x \in A \cap G$ & $y \in B \cap G$.~~

← Suppose every $x, y \in G$ can be joined by a polygonal curve in G .

Suppose G was disconnected & let (A, B) be a disconnection.

Let $x \in A \cap G, y \in B \cap G$. Let $z_0 = x, z_1, \dots, z_{n-1}, z_n = y$

be the endpoints of n line segments joining x to y .



Let z_k be the last endpoint lying in A : note $k \leq n-1$.

So $z_{k+1} \in B$. Define

$$A_1 = \{t \in \mathbb{R} : (1-t)z_k + tz_{k+1} \in A \cap G\}$$

$$B_1 = \{t \in \mathbb{R} : (1-t)z_k + tz_{k+1} \in B \cap G\}$$

Then (A_1, B_1) is a disconnection of $[0, 1]$, which is a contradiction. □

13. Complex Numbers

Read on your own.

∃ multiplication of complex nos.
ordering of \mathbb{C}