11. Heine-Borel Theorem

We start with the definition of a compact set.

Definition

A set $K \subseteq \mathbb{R}^p$ is compact if:

whenever K is contained in the union of any open sets G_α (α running through some index set I), then K must be contained in the union of finitely many of those G_α.

In other words:

$$K \subseteq \bigcup_{\alpha \in I} G_\alpha,\ G_\alpha \text{ open} \Rightarrow \exists\, \alpha_1, \dots, \alpha_n \text{ s.t. } K \subseteq \bigcup_{k=1}^{n} G_{\alpha_k}$$

Note

This is a conditional statement. It does not say that K is compact if it can be covered by finitely many open sets. Instead it says something much more involved: If you cover K by open sets, then no matter what this covering is, you can select finitely many of these open sets that still cover K:

Every cover of K by open sets has a finite subcover.

Again: not just some cover, but every possible open cover must have a finite subcover.

Example
Suppose that $K = \{x_1, \dots, x_n\}$ is a finite set.

Suppose that G_α, $\alpha \in I$ are any open sets such that $K \subseteq \bigcup_{\alpha \in I} G_\alpha$. Then $x_1 \in K \subseteq \bigcup_{\alpha \in I} G_\alpha$, so there is some α_1 s.t. $x_1 \in G_{\alpha_1}$. Likewise x_2 belongs to some G_{α_2}, etc. Hence $K \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$, i.e., K is covered by finitely many of the G_α, no matter what sets G_α we chose to cover K with.

Showing that a set is compact can be hard: you have to prove that every possible open cover has a finite subcover.

What do we have to do to prove that a set A is not compact? We have to prove that it's not true that every open cover has a finite subcover. To do that we only have to show that there exists one covering of A by open sets G_α s.t. if we only use finitely many of the G_α then A won't be covered.

Example: $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is not compact.

To prove this, we have to find a cover of A by open sets that no longer covers A if only finitely many of those open sets are used. There may be other covers that have finite subcovers, but that doesn't matter. For example, we can cover this set A with one open set:

$$A \subseteq (0,2).$$

This does not show A is compact & it does not show A is not compact.

Instead, set $G_1 = (\frac{1}{2}, 2)$ and $G_n = (\frac{1}{n+1}, \frac{1}{n-1})$ for $n \geq 2$. Then each G_n is open & $\frac{1}{n} \in G_n$ for each $n \in \mathbb{N}$, so

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq \bigcup_{n=1}^{\infty} G_n.$$

However, each point $\frac{1}{n}$ belongs to one and only one set G_k, namely G_n. Hence if we only use finitely many G_n, say $G_{n_1} \cup \cdots \cup G_{n_k}$, then we'll only cover the points $n_1, \dots, n_k$ and there will be infinitely many points in A that are not covered. Thus, <u>this particular</u> open cover $\{G_n\}_{n \in \mathbb{N}}$ <u>has no finite subcover</u>. Therefore A is compact.

<u>Notation</u>

If $E_\alpha, \alpha \in I$ are sets s.t.

$$A \subseteq \bigcup_{\alpha \in I} E_\alpha$$

then we call $\{E_\alpha\}_{\alpha \in I}$ a <u>cover</u> of A.

If each E_α is open, then it is an <u>open cover</u>.

If $J \subseteq I$ & $\{E_\alpha\}_{\alpha \in J}$ still covers A, then we call it a <u>subcover</u>. If J is finite then it is a <u>finite subcover</u>.

Example The interval $A = (0,1)$ is not compact.

Again we just have to find one cover of A by open sets that has no finite subcover. Consider

$$G_n = \left(\frac{1}{n}, 1\right).$$

Each $G_n \subseteq (0,1)$, so we certainly have $\bigcup_{n=1}^{\infty} G_n \subseteq (0,1)$. On the other hand, if $x \in (0,1)$ then $0 < x < 1$, so $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < x < 1$ (why?) Hence $x \in G_n$, so $(0,1) \subseteq \bigcup_{n=1}^{\infty} G_n$. Thus $\{G_n\}_{n \in \mathbb{N}}$ is an open cover of $(0,1)$. But if we only use finitely many G_n, say $G_{n_1}, \ldots, G_{n_k}$, then we cannot cover $(0,1)$.

Why? By reordering, we can take $n_1 < \cdots < n_k$. Then no x between 0 & $\frac{1}{n_k}$ will be covered by $G_{n_1}, \ldots, G_{n_k}$ (why?). We can't cover all of $(0,1)$ using finitely many G_n, so $(0,1)$ is not compact.

We will shortly see some better ways of dealing with compact sets in $\mathbb{R}^p$, but for now, use the definition of compactness to show the following.

Exercise

Suppose that K is a compact subset of $\mathbb{R}^p$.

Choose any $r>0$. Show $\exists$ finitely many points $x_1, \ldots, x_n \in \mathbb{R}^p$ s.t.

$$K \subseteq \bigcup_{k=1}^{n} B_r(x_k).$$

Hint: Consider all of the balls!

Next we show that in $\mathbb{R}^p$, a set is compact if & only if it is closed & bounded.

Beware: Compact, open, & closed sets make sense in much more general topological spaces, and it is not true in an arbitrary topological space that compact = closed & bounded!!

<u>Heine-Borel Theorem</u> Let $F \subseteq \mathbb{R}^p$ be given. Then:

$F \subseteq \mathbb{R}^p$ is compact $\iff$ F is closed & bounded.

<u>Proof:</u>

$\Rightarrow$ Assume F is compact.

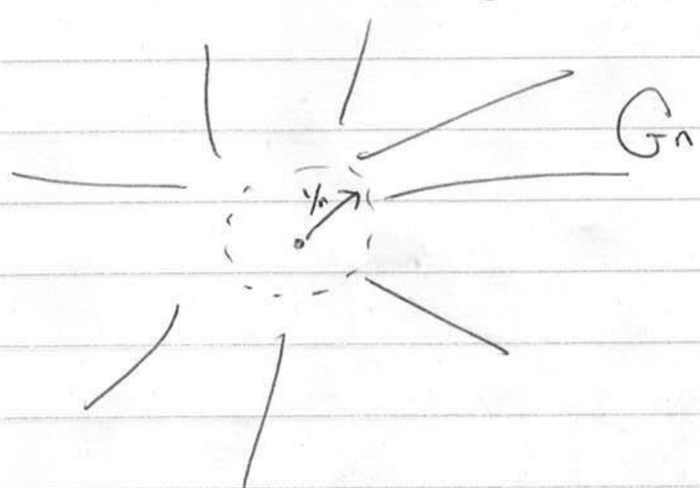
Let $G_n = B_n(0)$. Then $F \subseteq \mathbb{R}^p = \bigcup_{n \in \mathbb{N}} G_n$.

Since F is compact, it must be contained in a finite union of G_n. But F is contained in some ball $B_n(0)$, so is bounded.

Next, we show F is closed by showing that $\mathcal{C}(F)$ is open.

Let $x \in \mathcal{C}(F)$. Let $G_n = \{ y \in \mathbb{R}^n : \|x-y\| > \frac{1}{n} \}$.

$= \mathcal{C}(\overline{B_{1/n}(x)})$

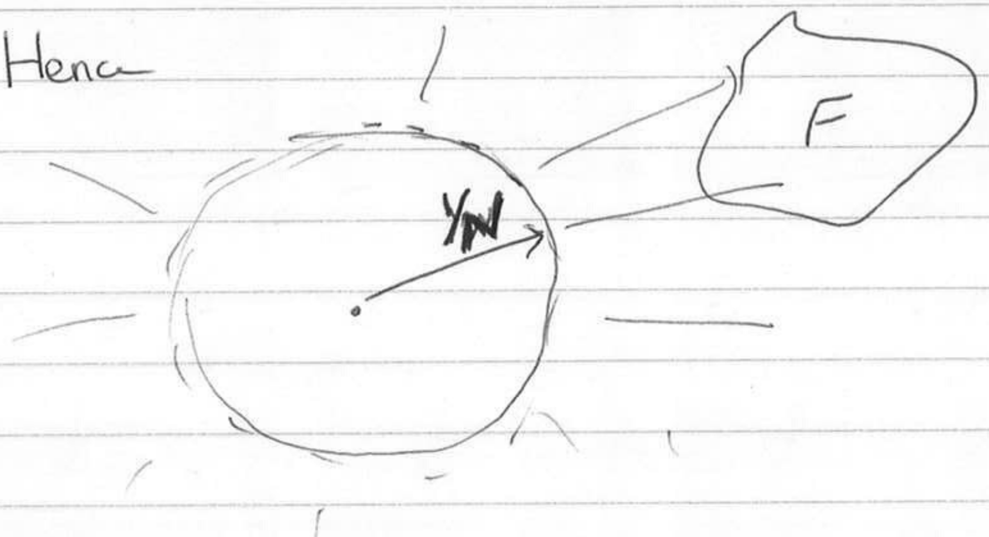


Note G_n is open, & $\bigcup_{n \in \mathbb{N}} G_n = \mathbb{R} \setminus \{0\} \supseteq F$.

Since F is compact, $F \subseteq G_1 \cup \cdots \cup G_N$

for some N. But $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_N$, so

$F \subseteq G_N$. Hence



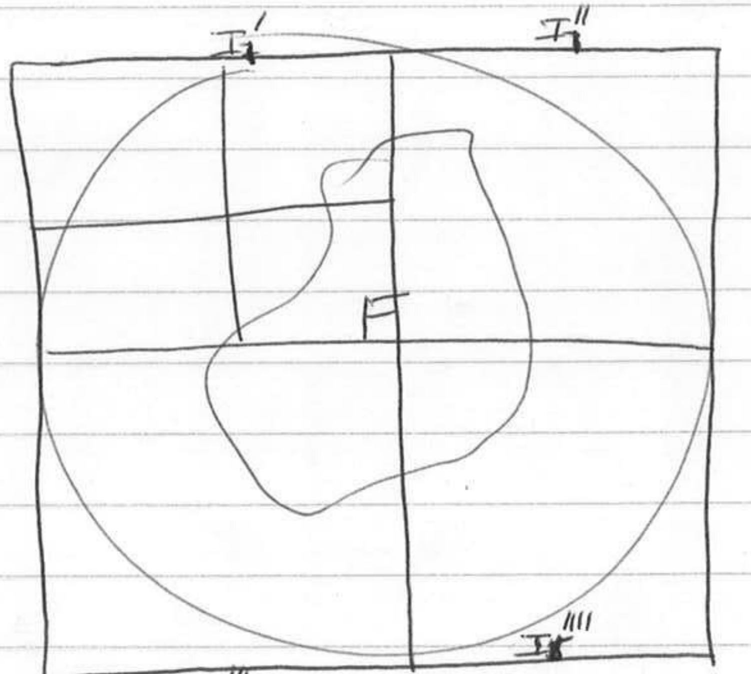
Thus $\bar{B}_{1/N}(x) \subseteq e(F)$ But then $x \in B_{1/2N}(x) \subseteq e(F)$
open ball

so $e(F)$ is open.

$\Leftarrow$ Assume F is closed & bounded.

Let $\{G_\alpha\}$ be any open cover of F (countable or uncountable). Suppose there is no finite subcover of $\{G_\alpha\}$.

Since F is bounded, $F \subseteq$ Cube I_1 for some r.



$\overline{B_r}(0) \subseteq I_1$

One of $F \cap I_1'$, $F \cap I_1''$, $F \cap I_1'''$, $F \cap I_1''''$ cannot be covered by finitely many G_α. Call this one I_2.

Repeat

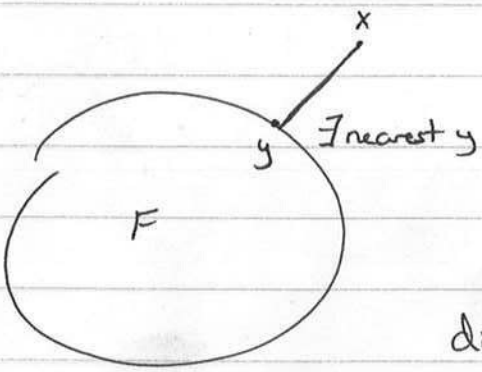
$I_1 \supseteq I_2 \supseteq \cdots$ Nested cells. $\exists\, y \in \bigcap I_n$

Note y is a cluster point of F. But F is closed, so $y \in F$.

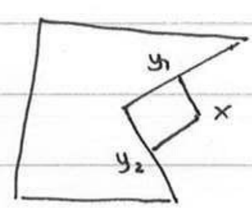
Hence $y \in G_\alpha$ for some α. But then $B_\varepsilon(y) \subseteq G_\alpha$ for some ε.

But $I_k \subseteq B_\varepsilon(y)$ for all $k \geq$ some k_0.

Hence $F \cap I_k$ is covered by the single set G_α — contradiction. $\blacksquare$



need not be unique



$\operatorname{dist}(x, F) = \|x - y\| > 0$

$\operatorname{dist}(x, u) = 0$

Cantor Intersection Theorem

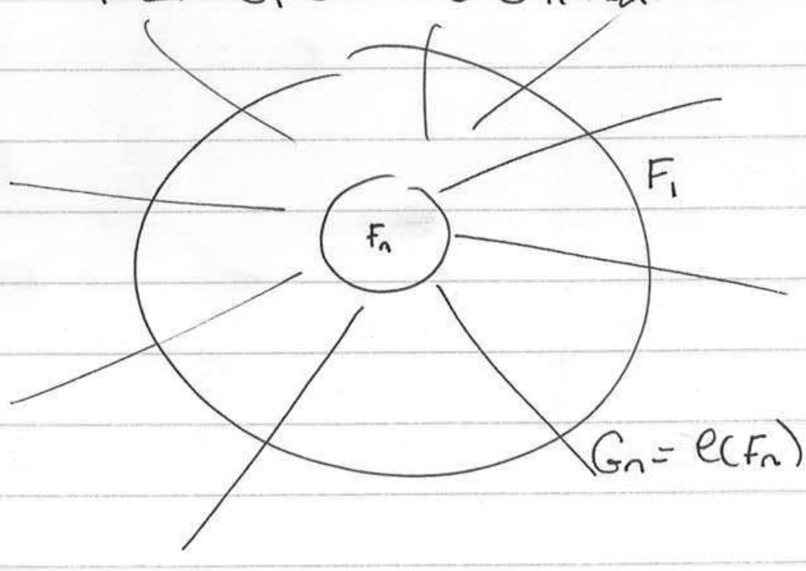
If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ are (closed, bounded) compact & nonempty, then $\bigcap F_n \neq \emptyset$.

Proof.

Let $G_n = \mathcal{C}(F_n)$. Suppose $\bigcap F_n = \emptyset$. Then $\bigcup G_n = \mathbb{R}^p$.

Hence $F_1 \subseteq \bigcup_n G_n$. But F_1 is closed & bounded, hence compact.

So $F_1 \subseteq G_1 \cup \cdots \cup G_n = G_n = \mathcal{C}(F_n)$.



$F_1 \cap F_n = \emptyset$

because

$F_1 \cap F_n \subset \mathcal{C}(F_n) \cap F_n = \emptyset$

Contradiction: $F_n \subseteq F_1$!

Example: If F_n aren't bounded, could have $\bigcap F_n = \emptyset$.

Let

$$F_n = \mathcal{C}(B_n(0)) = \{y \in \mathbb{R}^p : \|y\| \geq n\}.$$

Nested: $F_1 \supseteq F_2 \supseteq \cdots$ but $\bigcap F_n = \emptyset$

<u>Corollary</u>

Let $F \subseteq \mathbb{R}^p$ be closed, $F \neq \emptyset$. Let $x \notin F$.

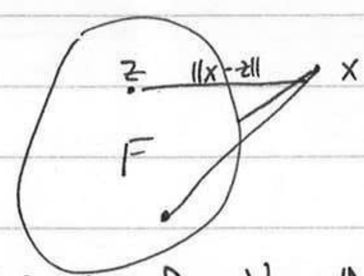
Then $\exists$ a point $y \in F$ that is nearest to x, i.e.

$$z \in F \implies \|z-x\| \geq \|z-y\|.$$

<u>Proof:</u>

Let

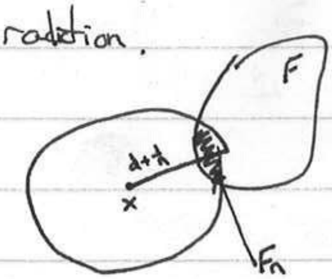
$$d = \inf\{\|z-x\| : z \in F\} \overset{= \operatorname{dist}(x,F)}{}.$$



We claim that $d > 0$. Suppose $d = 0$. Then by def. of inf, $\forall n \in \mathbb{N}$,

$\exists z_n \in F$ s.t. $\underset{\substack{\uparrow \\ \text{because } x \notin F}}{0 <} \|z_n - x\| < \frac{1}{n}$. Hence x is a cluster point of F.

But F is closed, so this implies $x \in F$, a contradiction.



Let $F_n = \overline{B}_{d+\frac{1}{n}}(x) \cap F \quad F_1 \supseteq F_2 \supseteq \dots$

By Cantor Intersection, $\exists y \in \bigcap_{n=1}^{\infty} F_n$. $\|y-x\| \leq d + \frac{1}{n} \ \forall n$

Note $y \in F$. So $\|y-x\| \leq d$.

But $d = \inf\{\|x-z\| : z \in F\}$ so $\|y-x\| \leq \|x-z\| \quad \forall z \in F$ ∎

<u>Note</u>: There can be more than one closest point