10. Nested Cells & Bolzano-Weierstrass

Definition

A point $x \in \mathbb{R}^p$ is a cluster point (or accumulation point, or limit point) of $A \subseteq \mathbb{R}^p$ if

$\forall$ neighborhood N of x, $\exists\, y \in N \cap A$ s.t. $y \neq x$.

That is, to be a cluster point, every neighborhood of x has to have a point of A other than x itself.

This is somewhat similar to, but different than boundary points!

As with anything involving neighborhoods, it really all comes down to questions about balls.

Exercise

Given $A \subseteq \mathbb{R}^p$ & $x \in \mathbb{R}^p$, prove TFAE.

a. x is a cluster point of A.

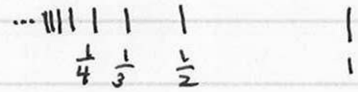
b. $\forall n \in \mathbb{N}$ $\exists\, y_n \in A$ s.t. $0 < \|x - y_n\| < \frac{1}{n}$.

(So points of A really do "cluster" about x)

Example $A = \{x\}$ has no cluster points, but it does have a boundary point (x itself)

Exercise

Consider $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$



a. Prove that $x = 0$ is the only cluster point of A.

b. Prove that $\partial A = A \cup \{0\}$.

Exercise

a. If $x \notin A$ but $x \in \partial A$, then x is a cluster point of A.

b. Give examples showing that if $x \in A$ and $x \in \partial A$, then x may or may not be a cluster point of A.

c. Show that all interior points are cluster points.

d. Show that no exterior points are cluster points.

We can characterize closed sets by their cluster points, just as we did for boundary points.

Exercise

Let $F \subseteq \mathbb{R}^p$ be given. Then

F is closed $\iff$ F contains all its cluster points.

Sketch for $\Rightarrow$

Suppose F is closed & x is a cluster point.

If $x \notin F$, show that x must be an exterior point, and that this implies it is not a cluster point.

Definition

A set $A \subseteq \mathbb{R}^p$ is bounded if it is contained in some open ball $B_r(x)$.

Exercise

Show A is bounded if & only if $\exists R > 0$ s.t. $\|x\| < R \; \forall x \in A$, or, in other words, $A \subseteq B_R(0)$.

Bolzano - Weierstrass Theorem

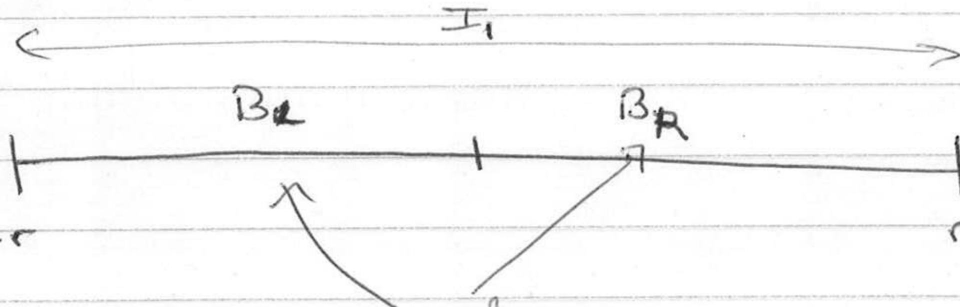
Every bounded infinite subset of $\mathbb{R}^p$ has a cluster point.

Proof:

$p=1$.

Let $B \subseteq \mathbb{R}$ be an infinite bounded set.

Then $B \subseteq I_1 = [-r, r]$ for some r. Let y_1 = one point in I_1



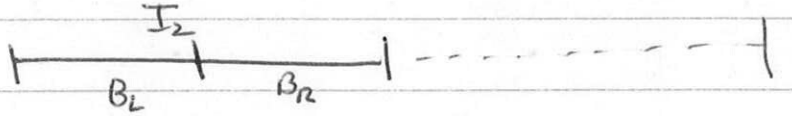
one of these becomes I_2

Let B_L = left half of I_1, B_R = right half of I_1
$[-r, 0]$ $[0, r]$

Either B_L contains ∞ many points of B
or B_R " " " " (or both).

Let I_2 = one with ∞ many points of B.
y_2 = one point of B in I_2, $\neq y_1$

Repeat.



One half of I_2 contains ∞ many points. of B.

I_3 = one half of I_2 that contains ∞ many points of B

y_3 = one point of B in I_3, $\neq$ y_1, y_2.

Etc.

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ nested cells.

$\exists \; y \in \bigcap_{n=1}^{\infty} I_n$ Nested cells property.

Note $y \in I_1,\ y_1 \in I_1 \Rightarrow \|y - y_1\| < 2r$

$y \in I_2,\ y_2 \in I_2 \Rightarrow \|y - y_2\| < r$

$\vdots$

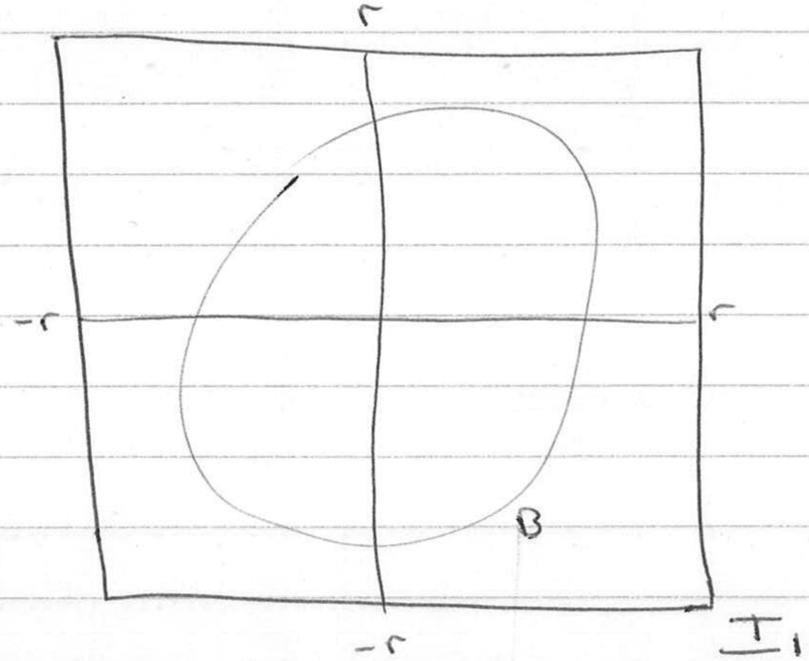
$y \in I_n,\ y_n \in I_n \Rightarrow \|y - y_n\| < r/2^{n-2}$

So $\lim_{n \to \infty} \|y - y_n\| = 0$.

And y can only equal at most one y_n.

So y is a cluster point of B. $\boxed{B}$

$\underline{p=2}$.



One subsquare becomes I_2, etc.