Real Analysis I

<u>Goal</u>: Prove basic mathematics of numbers & functions.

Set theory axioms

1. "There is a set"
2. If A, B are sets, then $A \cup B$ is a set

$\vdots$

Assume true

$\Downarrow$

Integers exist

Rational numbers exist

Real numbers exist

Properties of these things

These follow logically

We Assume true (although we'll look at some of this later)

$\Downarrow$

More advanced properties (our goal)

Set Theory Review

Read Section 1

A <u>set</u> is a collection of objects.

The objects are called the <u>elements</u> of the set.

<u>Examples</u>

$$S = \{2, 3, 5, 7, 11, 13\} \qquad T = \{5, 7, 11\} \qquad U = \{2, 4, 6\}$$

$$3 \in S, \; 4 \notin S, \; \pi \notin S \qquad 5 \in T$$

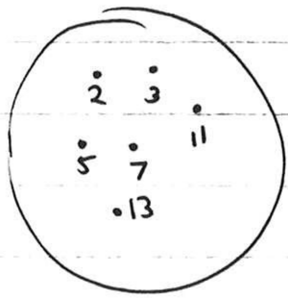
<u>Union</u> $T \cup U = \{2, 4, 5, 6, 7, 11\}$ (order <u>not</u> important / duplicates not allowed)

<u>Intersection:</u> $S \cap U = \{2\}$ Note $\{2\} \neq 2$

<u>Emptyset:</u> $T \cap U = \emptyset = \{\}$ No elements

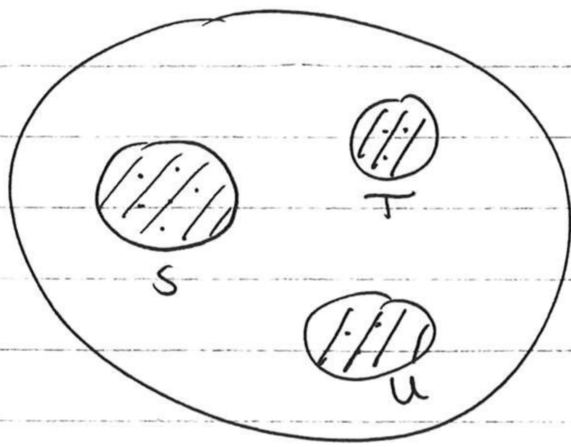
<u>More sets</u> $E = \{S, T, U\}$ has three elements, each of which are <u>themselves</u> sets.

$$F = \{\emptyset, \{2\}, E\} \qquad \text{3 elements}$$



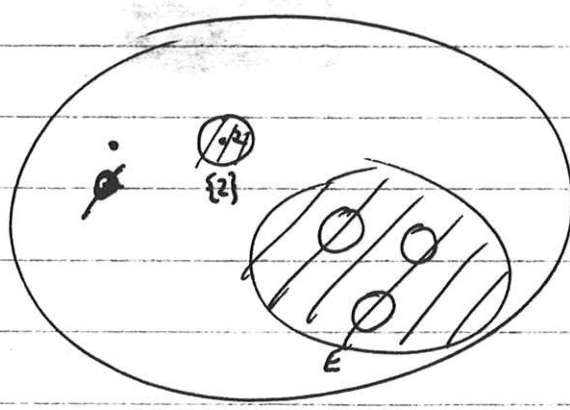
S

has 6 objects



E

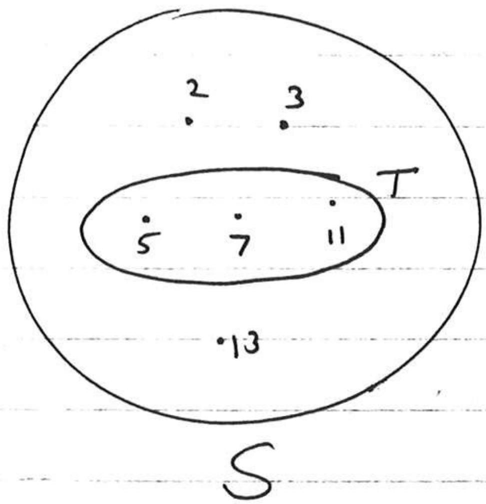
3 objects, each object is itself a set



F

3 objects

Compare S & T:



Every element of T is also an element of S.

Write $T \subseteq S$

A is a <u>subset</u> of B if every element of A is also an element of B. Write $A \subseteq B$.

Note: A <u>could</u> <u>be</u> equal to B, $A \subseteq A$ is true.

Note: $\emptyset \subseteq A$ is always true

$\boxed{\text{Bartle: } \subseteq}$

Does <u>not</u> mean $\emptyset \in A$!

<u>Ex.</u> $A = \{\emptyset, 1, 2, 5\}$ has $\emptyset \in A$

$B = \{1, 2, 5\}$ $\emptyset \notin B$

But $\emptyset \subseteq A$ & $\emptyset \subseteq B$, $B \subseteq A$.

<u>Examples:</u>

$\emptyset \in F$ $\qquad\qquad$ $\{\emptyset, E\} \subseteq F$

$\{2\} \in F$

$E \in F$

$2 \notin F$ $\qquad\qquad$ $\{2\} \not\subseteq F$ $\qquad\qquad$ $\{\{2\}\} \subseteq F$

<u>Equality</u> $\quad A = B$ means A & B have <u>exactly</u> the same elements.

$A = B \iff$ Every element of A belongs to B
AND " " " B " " A.

$$A = B \iff \left(A \subseteq B \text{ AND } B \subseteq A. \right)$$

Specifying sets by properties

We often define sets by specifying what property its elements must have in order to belong to the set:

generic form: $S = \{x : x \text{ has property } P\}$.

Ex: $A = \{x : x \text{ is an apple}\}$.

Then $x \in S$ <u>means exactly</u> that x has property P.

Ex. $x \in A \iff x$ is an apple.

<u>Ex:</u> Let $I = \{x \in \mathbb{R} : 2 \leq x \leq 3\}$.

If $x \in I$ then it must be true that x is between 2 & 3.

Conversely, if x is a real no. between 2 & 3 then $x \in I$.

Special Sets

$\mathbb{N} = \{1, 2, 3, \ldots\}$ set of natural numbers

$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ Set of integers

$\mathbb{R} = \{x : x \text{ is a real number}\}$ set of real numbers

Visualize:

each point is one element of the set

$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z} \ \& \ n \neq 0\}$ set of rational numbers.

$\mathbb{C} = \{z : z \text{ is a complex number}\}$

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

These are infinite sets

Russel's Paradox: Def. of set is naive.

Q. Is $\mathcal{S} = \{S : S \text{ is a set}\}$ a set?

$\mathcal{S}$ has the unusual property $\mathcal{S} \in \mathcal{S}$

$G = \{S : S \text{ is a set } \& \ S \notin S\}$

$B = \{S : S \text{ is a set } \& \ S \in S\}$

Q. Is $G \in G$?

Analyze:

If G is a set & $G \in G$ then $G \notin G$

If G is a set & $G \notin G$ then $G \in G$.

G cannot be a set

There are many basic properties of sets.

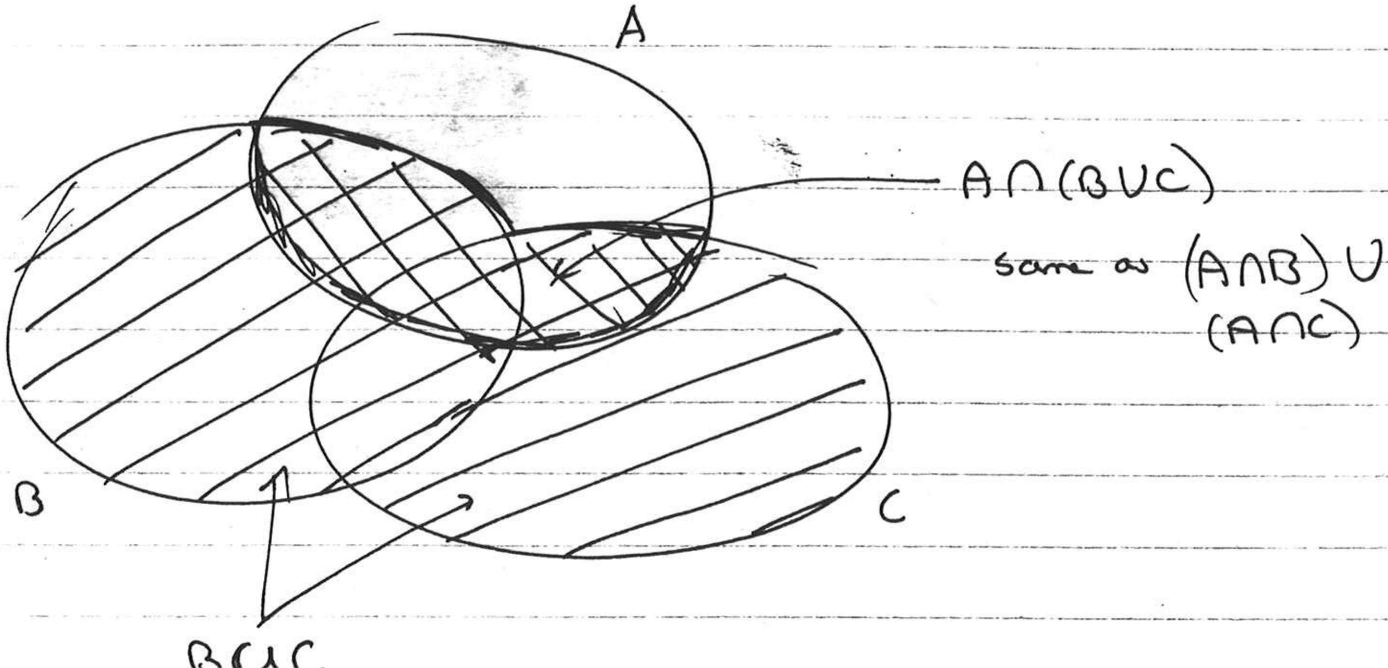
<u>Theorem</u> $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

<u>Proof:</u> Write in <u>English</u> using <u>complete</u> sentences.

Suppose $x \in A \cap (B \cup C)$. This means that $x \in A$ and $x \in B \cup C$. By def. of union, this implies either $x \in B$ or $x \in C$ (or both). Hence we either have $x \in A \cap B$ or $x \in A \cap C$ (or both). Therefore $x \in (A \cap B) \cup (A \cap C)$. This shows that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

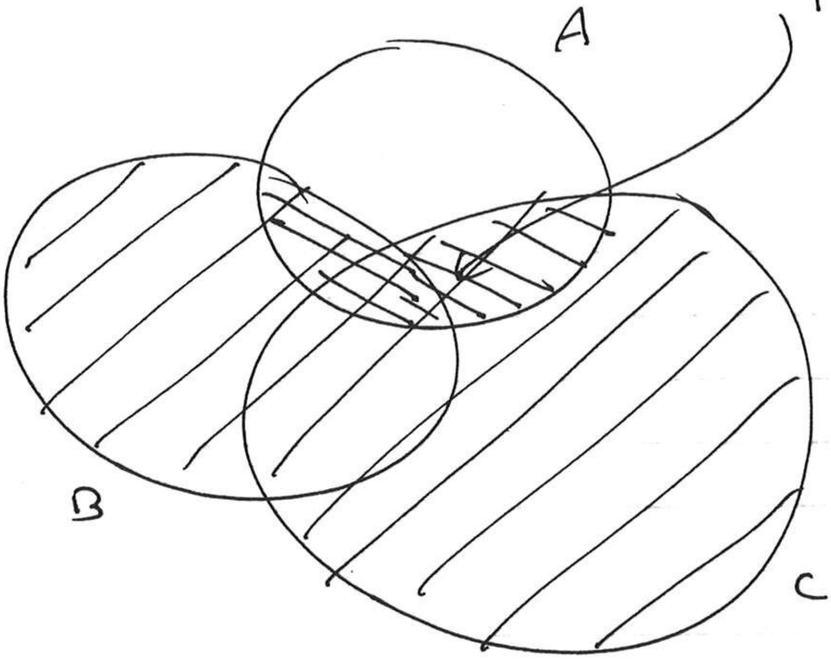
EXERCISE: To complete the proof, show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. ∎

<u>Motivation</u> by picture (not proof!)



Picture can motivate proofs.

$A \cap (B \cup C)$
same as $(A \cap B) \cup (A \cap C)$.



$/// = B \cup C$

Pictures are <u>not</u> proofs.

<u>Complements</u>

$$A \setminus B = \{x : x \in A, x \notin B\}$$

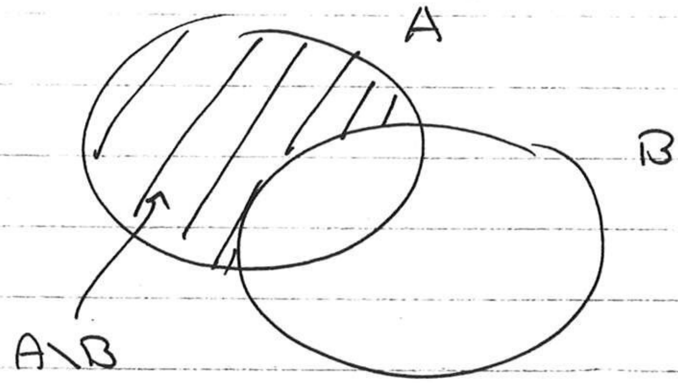
If A is understood, write $\mathcal{C}(B) = A \setminus B$.

<u>Ex</u>. $\mathbb{R}$ = set of real numbers, understood to be the context
$\mathbb{Q}$ = set of rationals

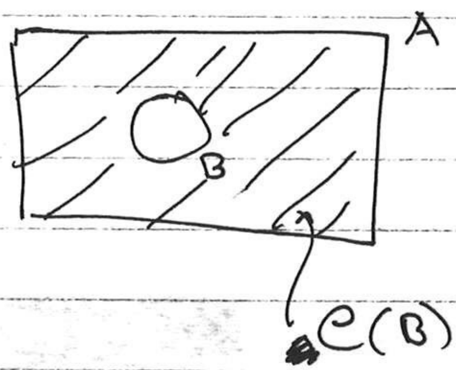
$\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ = set of irrationals

Complements

$$A \setminus B = \{x : x \in A \text{ but } x \notin B\}$$



If A is understood, write $\cancel{A \setminus B}$ $\mathcal{C}(A)$ or A^C



Ex: $A = \mathbb{R}$ is understood.

$$\mathcal{C}(\mathbb{Q}) = \{x : x \text{ is irrational}\}$$

Cartesian Product

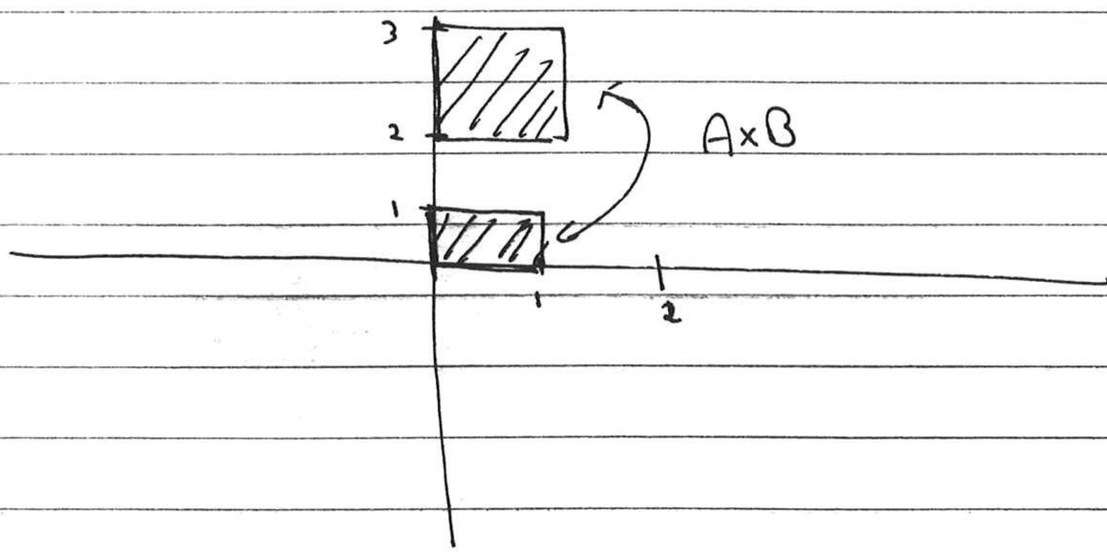
$$A \times B = \{ (a,b) : a \in A,\ b \in B \}$$

Ex: $A = \{1,2\}$ $\quad B = \{3,4,5\}$

$$A \times B = \{ (1,3), (1,4), (1,5), (2,3), (2,4), (2,5) \}$$

Ex: $A = [0,1] = \{ x \in \mathbb{R} : 0 \le x \le 1 \}$

$B = [0,1] \cup [2,3] = \{ x \in \mathbb{R} : 0 \le x \le 1 \text{ OR } 2 \le x \le 3 \}$



$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x,y) : x, y \in \mathbb{R} \} \quad \text{Cartesian Plane}$$