MATH 4317 Real Analysis I

SOME RECOMMENDED PROBLEMS WITH SOLUTIONS

Here are a few practice problems with solutions. Try to work these WITHOUT looking at the solutions! After you write your own solution, you can compare to my solution. Your solution does not need to be identical—there are often many ways to solve a problem—but it does need to be CORRECT.

Problem 1 M. If B_1 and B_2 are subsets of B and if $B = B_1 \cup B_2$, then

$$A \times B = (A \times B_1) \cup (A \times B_2).$$

Solution

Suppose that $x \in A \times B$. Then $x = (a, b)$ for some $a \in A$ and some $b \in B$. Since $b \in B = B_1 \cup B_2$, we have that b is either in B_1 or it is in B_2 (or both). Therefore either $(a, b) \in A \times B_1$ or $(a, b) \in A \times B_2$, so $x = (a, b) \in (A \times B_1) \cup (A \times B_2)$. This shows that $A \times B \subseteq (A \times B_1) \cup (A \times B_2)$.

For the reverse inclusion, suppose that $x \in (A \times B_1) \cup (A \times B_2)$. Then either $x \in A \times B_1$ or $x \in A \times B_2$. If $x \in A \times B_1$ then $x = (a, b)$ with $a \in A$ and $b \in B_1$. But $B_1 \subseteq B$, so $b \in B$ and therefore $x = (a, b) \in A \times B$. On the other hand, if $x \in A \times B_2$ then $x = (a, b)$ with $a \in A$ and $b \in B_2$. But $B_2 \subseteq B$, so $b \in B$ and therefore $x = (a, b) \in A \times B$. Thus in any case we have $x \in A \times B$. This shows that $(A \times B_1) \cup (A \times B_2) \subseteq A \times B$ and completes the proof. □

Problem 2 G. Let f and g be functions and suppose that $(g \circ f)(x) = x$ for all x in $D(f)$. Show that f is an injection and that $R(f) \subseteq D(g)$ and $R(g) \supseteq D(f)$.

Solution

First we show that f is an injection. Suppose that $x, y \in D(f)$ were such that $f(x) = f(y)$. Then, by definition of composition and by the hypotheses on f and g, we have

$$x = (g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y) = y.$$

Thus $f(x) = f(y)$ implies $x = y$, and therefore f is an injection.

Now we show that $R(f) \subseteq D(g)$. Here we are trying to show that one set is contained in another set, so we must show that every element of the first set is also an element of the second set. Therefore, suppose that $y \in R(f)$. Then, by definition, $y = f(x)$ for some $x \in D(f)$.

Solution

a. Since r is rational, we know that $r = m/n$ for some integers m and n with $n \neq 0$. Suppose that $r + \xi = p/q$ for some integers p and q with $q \neq 0$. Then $\xi = p/q - r = p/q - m/n = (np - mq)/(nq)$, and therefore ξ is rational. We have thus proved that $r + \xi$ rational implies ξ rational, which is the contrapositive of the statement ξ irrational implies $r + \xi$ rational. Note that we really didn't need the assumption $r \neq 0$ in this part, it's still true even if $r = 0$.

b. Again, since r is a NONZERO rational, we know that $r = m/n$ for some integers m and n with $m \neq 0$ AND $n \neq 0$. Suppose that $r\xi = p/q$ for some integers p and q with $q \neq 0$. Then $\xi = p/(qr) = (np)/(qm)$, so ξ is rational since np and qm are integers with $qm \neq 0$. Thus we have proved the contrapositive statement $r\xi$ rational implies ξ rational. Note that we did use the assumption $r \neq 0$ in this part. In fact, the problem would be false without this assumption, since $0\xi = 0$ is rational for every ξ. $\square$

Problem 5 C. If $a > -1$, $a \in \mathbf{R}$, show that $(1+a)^n \geq 1 + na$ for all $n \in \mathbf{N}$. This inequality is called *Bernoulli's Inequality.* (Hint: use mathematical induction.)

Solution

The hint suggests using mathematical induction, so let use it. Mathematical induction says that we just have to prove the following two things: (a) Base step: Show the statement is true for the specific case $n = 1$, and (b) Inductive step: Show that IF the statement is true for some n THEN it is also true for $n + 1$.

Base step $n = 1$. We have to show that $(1 + a)^1 \geq 1 + 1 \cdot a$. This is trivial because $(1 + a)^1 = 1 + a$ and $1 + 1 \cdot a = 1 + a$. Hence, not only is it true that $(1 + a)^1 \geq 1 + 1 \cdot a$, but it is in fact true that $(1 + a)^1 = 1 + 1 \cdot a$ (although for general n we cannot put in an equality, only the inequality).

Inductive step. ASSUME that the statement is true for some $n \geq 1$. We then have to somehow show that the statement is also true for $n + 1$. Here is the reasoning:

$$
\begin{aligned}
(1+a)^{n+1} &= (1+a)\,(1+a)^n && \text{by definition} \\
&\geq (1+a)\,(1+na) && \text{by our assumption that the statement is true for } n \\
&= 1 + (n+1)a + na^2 && \text{algebra} \\
&\geq 1 + (n+1)a && \text{since } na^2 \geq 0.
\end{aligned}
$$

Therefore the statement is true for $n + 1$ as well. $\square$

yields an inner product on $\mathbf{R}^2$. Generalize this to $\mathbf{R}^p$.

NOTE: This is an example of a *weighted inner product.* It allows some directions in the coordinate system to be weighted more heavily than others.

Solution

We have to show that properties (i)–(v) of the definition of an inner product in Definition 8.3 are satisfied.

(i) Given $x = (x_1, x_2) \in \mathbf{R}^2$, we have

$$x \cdot x \;=\; (x_1, x_2) \cdot (x_1, x_2) \;=\; x_1 x_1 w_1 + x_2 x_2 w_2 \;=\; x_1^2 w_1 + x_2^2 w_2. \tag{2}$$

Since x_1^2, $x_2^2 \geq 0$ and w_1, $w_2 > 0$, it follows that $x \cdot x \geq 0$.

(ii) Suppose that $x \cdot x = 0$. Then, as in equation (2), we have $x_1^2 w_1 + x_2^2 w_2 = 0$. But since every term in this sum is positive, we must then have $x_1^2 w_1 = x_2^2 w_2 = 0$. Since w_1, $w_2 \neq 0$, it therefore follows that $x_1^2 = x_2^2 = 0$, which implies $x = (x_1, x_2) = (0, 0) = 0$.

Conversely, suppose that $x = 0$. Then $x \cdot x = 0^2 \cdot w_1 + 0^2 \cdot w_2 = 0$.

(iii) If $x = (x_1, x_1) \in \mathbf{R}^2$ and $y = (y_1, y_2) \in \mathbf{R}^2$ then

$$x \cdot y \;=\; (x_1, x_2) \cdot (y_1, y_2) \;=\; x_1 y_1 w_1 + x_2 y_2 w_2 \;=\; x_2 y_2 w_2 + x_1 y_1 w_1 \;=\; (y_1, y_2) \cdot (x_1, x_2) \;=\; y \cdot x.$$

(iv) If $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in \mathbf{R}^2$ then

$$\begin{aligned} x \cdot (y + z) \;=\; (x_1, x_2) \cdot (y_1 + z_1, y_2 + z_2) \;&=\; x_1(y_1 + z_1)w_1 + x_2(y_2 + z_2)w_2 \\ &=\; x_1 y_1 w_1 + x_2 y_2 w_2 \;+\; x_1 z_1 w_1 + x_2 z_2 w_2 \\ &=\; (x_1, x_2) \cdot (y_1, y_2) \;+\; (x_1, x_2) \cdot (z_1, z_2) \\ &=\; x \cdot y + x \cdot z. \end{aligned}$$

This fact, combined with the commutative property (iii), can easily be shown to imply that we also have $(x + y) \cdot z = x \cdot z + y \cdot z$.

(v) If $a \in \mathbf{R}$ and $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{R}$, then

$$(ax) \cdot y \;=\; (ax_1, ax_2) \cdot (y_1, y_2) \;=\; (ax_1)y_1 w_1 + (ax_2)y_2 w_2 \;=\; a(x_1 y_1 w_1 + x_2 y_2 w_2) \;=\; a(x \cdot y).$$

Combining this with the commutative property (iii), we see that we also have $x \cdot (ay) = a\,(x \cdot y)$.

The fact that this definition of $x \cdot y$ satisfies properties (i)–(v) means that it is an inner product for $\mathbf{R}^2$. In particular, each choice of w_1 and w_2 leads to a distinct inner product for closest to x, but then we could choose n so that r/n was closer than that point). Hence $\{y_n\}_{n\in\mathbf{N}}$ is an infinite set of points that lie in both A and N.

$\Leftarrow$. Suppose that every neighborhood of x contains infinitely many points of A. Let N be any neighborhood of x. Then we know that there exist infinitely many points $\{y_n\}_{n\in\mathbf{N}}$ that lie in both A and N. Hence at least one of these points, say $y = y_n$, must be different from x. But then $y \in A \cap N$ with $y \neq x$, so x satisfies the requirements of a cluster point. □

Problem 11 B. Prove directly that the entire space $\mathbf{R}^2$ is not compact.

Solution

Of course, $\mathbf{R}^p$ is not bounded, so it can't be compact by the Heine–Borel Theorem. But the point is: can you show *directly from the definition* of compact set that $\mathbf{R}^p$ isn't compact?

Here is one method. Let $U_n = B_n(0)$, i.e., U_n is the open ball of radius n centered at the origin. Note that $U_1 \subseteq U_2 \subseteq \cdots$. Further, we certainly have $\mathbf{R}^p \subseteq \bigcup_{n\in\mathbf{N}} U_n$. However, if we select only *finitely* many sets out of this open cover, say $U_{k_1}, \dots, U_{k_n}$ with $k_1 < \cdots < k_n$, then the largest one, U_{k_n}, contains all the balls $U_{k_1}, \dots, U_{k_n}$, so $U_{k_1} \cup \cdots \cup U_{k_n} = U_{k_n} \neq \mathbf{R}^p$. Hence $\{U_n : n \in \mathbf{N}\}$ is one cover of $\mathbf{R}^p$ by open sets that contains no finite subcover of $\mathbf{R}^p$. Therefore $\mathbf{R}^p$ isn't compact. □

Problem 12 A. If A and B are connected subsets of $\mathbf{R}^p$, give examples to show that $A \cup B$, $A \cap B$, $A \backslash B$ can be either connected or disconnected.

Solution

Let $A = [0, 2]$ and $B = [1, 3]$. Then $A \cup B = [0, 3]$, $A \cap B = [2, 3]$, and $A \backslash B = [0, 1)$ are all connected.

On the other hand, let $A = [0, 1]$ and $B = [2, 3]$. Then $A \cup B = [0, 1] \cup [2, 3]$ is disconnected (what is one disconnection?). If $A = [0, 3]$ and $B = [1, 2]$, then $A \backslash B = [0, 1) \cup (2, 3]$ is disconnected. However, in one dimension, the connected sets are just the intervals, and there is no example of connected intervals A, B such that $A \cap B$ not connected. But if we move to $\mathbf{R}^2$, then we can find such sets. For example, let A be the top half of a semicircle of radius 1 and let B be the bottom half, i.e.,

$$A = \{(x, y) : x^2 + y^2 = 1,\ y \geq 0\}, \qquad B = \{(x, y) : x^2 + y^2 = 1,\ y \leq 0\}.$$

Then $A \cap B = \{(-1, 0),\ (0, 1)\}$ contains only two points, hence is disconnected.

It is easy to create analogues of these examples in higher dimensions as well. □

Therefore, $x_n^{1/n} = e^{\ln x_n^{1/n}} \to e^0 = 1$.

Another example of a divergent sequence (x_n) whose nth roots $(x_n^{1/n})$ do converge to 1 is $(x_n) = (1, 2, 1, 2, 1, 2, \dots)$. This sequence doesn't converge, but $(1, 2^{1/2}, 1, 2^{1/4}, 1, 2^{1/6}, \dots)$ does converge to 1. $\square$

Problem 15 D. If X and Y are sequences in $\mathbf{R}^p$ and if $X + Y$ converges, do X and Y converge and have $\lim (X + Y) = \lim X + \lim Y$?

Solution

A theorem in the books states that if X and Y do converge, then $X + Y$ converges as well. The point of this problem is that the *converse* of this statement doesn't need to be true: just because $X + Y$ converges, it need not be true that X and Y themselves converge. For example, consider the sequences of numbers given by $x_n = (-1)^n$ and $y_n = -(-1)^n$. Neither $X = (x_n)$ nor $Y = (y_n)$ converge, but $X + Y = (x_n + y_n) = (0, 0, 0, \dots)$ does converge. $\square$

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Problem 16 H. Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbf{R}^p$ and let $Z = (z_n)$ be the "shuffled" sequence defined by

$$z_1 = x_1, \quad z_2 = y_1, \quad \dots, z_{2n} = x_n, \quad z_{2n+1} = y_n, \dots.$$

Is it true that Z is convergent if and only if X and Y are convergent and $\lim X = \lim Y$?

Solution

Yes, it's true.

$\Rightarrow$. Suppose that Z is convergent. Since X and Y are both subsequences of Z, they must then be convergent and must converge to the same limit as Z, i.e., $\lim X = \lim Z = \lim Y$.

$\Leftarrow$. Suppose X and Y both converge and that $\lim X = \lim Y = x$. We claim that Z also converges to x. To see this, choose $\varepsilon > 0$. Then:

$$\exists\, N_1 > 0 \text{ such that } \quad n \geq N_1 \implies \|x - x_n\| < \varepsilon,$$

$$\exists\, N_2 > 0 \text{ such that } \quad n \geq N_2 \implies \|x - y_n\| < \varepsilon.$$

Let $N = \max 2N_1, 2N_2 - 1$. Suppose $n \geq N$. If n is even, say $n = 2k$, then $k \geq N_1$, so $\|z_n - x\| = \|x_k - x\| < \varepsilon$. On the other hand, if n is odd, say $n = 2j - 1$, then $j \geq N_2$, so $\|z_n - x\| = \|y_j - x\| < \varepsilon$. In any case, $\|z_n - x\| < \varepsilon$ for all $n \geq N$, so $z_n \to x$. $\square$

Problem 16 I. Show directly that the following are Cauchy sequences.

(a) $(1/n)$.

Solution

The sequence is $x_n = 1/n$. So, if $m > n$ then

$$|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| = \frac{1}{n} - \frac{1}{m} \leq \frac{1}{n}.$$

Show that $\lim(g_n(x)) = 0$ for all $x > 0$.

Solution

Before you do anything—draw a picture!!

Let $x > 0$ be fixed. Hence, then there exists $N > 0$ such that $1/N < x$. Therefore, if $n \geq N$ then $1/n \leq 1/N < x$, so $g_n(x) = 1/(nx)$. But $r = 1/x$ is just a fixed, positive number, so for $n \geq N$ we have $g_n(x) = r/n \to 0$.

This proof is rigorous enough, but if we like, we could also write it out with explicit ε's. To do this, $\varepsilon > 0$. Then let $N > 0$ be large enough so that both $1/N < x$ and $1/N < \varepsilon x$. Then, $n \geq N$ implies $1/n \leq 1/N < x$, so $g_n(x) = 1/(nx) \leq 1/(Nx) < \varepsilon$. This shows that $g_n(x) \to 0$ for each individual x.

NOTE: Since $g_n(1/n) = 1$ for every n, it is NOT true that $g_n \to 0$ uniformly on $[0, \infty)$. For uniform convergence, we would have to have $\|g_n\|_\infty \to 0$, but we don't have this because $\|g_n\|_\infty = 1$ for every n. On the other hand, you will show in problem 17L that the convergence *is* uniform if we restrict the domain to $[c, \infty)$ with $c > 0$. $\square$

Problem 17 I. Suppose that (x_n) is a convergent sequence of points which lies, together with its limit x, in a set $D \subseteq \mathbf{R}^p$. Suppose that (f_n) converges on D to the function f. Is it true that $f(x) = \lim f_n(x_n)$?

Solution

No. For example, think about the very simplest case of convergent functions, i.e., when every f_n is just f. Then $\lim f_n(x_n) = \lim f(x_n)$. Does this have to equal $f(x)$? The answer is no if f has a discontinuity at x. For example, define

$$f_n(x) = f(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then let $x_n = 1/n$ and $x = 0$. Then $f_n(x_n) = f(x_n) = f(1/n) = 1$ for every n, but $f(x) = f(0) = 0$. $\square$

Problem 17 L. Show that the convergence in Exercise 17B is not uniform on the domain $x \geq 0$, but that it is uniform on a set $x \geq c$, where $c > 0$.

Solution

I already pointed out that the convergence is not uniform on the domain $[0, \infty)$, because $g_n(1/n) = 1$ for every n. But suppose that we consider instead the domain $D = [c, \infty)$, where

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Problem. Let (x_n) and (y_n) be bounded sequences of nonnegative real numbers. Let $x^* = \limsup\,(x_n)$. Prove that

$$\limsup_{n\to\infty} x_n y_n \;\le\; \Big(\limsup_{n\to\infty} x_n\Big)\Big(\limsup_{n\to\infty} y_n\Big).$$

Solution

Set

$$x^* \;=\; \limsup_{n\to\infty} x_n, \qquad y^* \;=\; \limsup_{n\to\infty} y_n, \qquad z^* \;=\; \limsup_{n\to\infty} x_n y_n.$$

We want to show that $z^* \le x^* y^*$. For $m \in \mathbf{N}$ define

$$u_m \;=\; \sup_{n\ge m} x_n, \qquad v_m \;=\; \sup_{n\ge m} y_n, \qquad w_m \;=\; \sup_{n\ge m} x_n y_n.$$

By one of the equivalent characterizations of limsup, we then have that

$$x^* \;=\; \lim_{m\to\infty} u_m, \quad y^* \;=\; \lim_{m\to\infty} v_m, \quad z^* \;=\; \lim_{m\to\infty} w_m.$$

We have the following inequalities for each m (WHY?):

$$w_m \;\le\; u_m v_m.$$

Hence

$$z^* \;=\; \lim_{m\to\infty} w_m \;\le\; \lim_{m\to\infty} u_m v_m \;=\; \Big(\lim_{m\to\infty} u_m\Big)\Big(\lim_{m\to\infty} v_m\Big) \;=\; x^* y^*.$$

Question: Must we actually have equality, or does there exist an example where equality does not hold? $\square$

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Problem 20 J. Let h be defined for $x \neq 0$, $x \in \mathbf{R}$, by

$$h(x) = \sin(\tfrac{1}{x}), \qquad x \neq 0.$$

Show that no matter how h is defined at $x = 0$, it will be discontinuous at $x = 0$.

Solution

Suppose that we were able to define $h(0)$ so that h was continuous at $x = 0$. Consider the sequences $x_n = 1/(2\pi n)$ and $y_n = 1/(\frac{\pi}{2} + 2\pi n)$. We have $x_n \to 0$ and $y_n \to 0$, but $h(x_n) = 0$ for every n while $h(y_n) = 1$ for every n, which contradicts that fact that we must have both $0 = h(x_n) \to h(0)$ and $1 = h(y_n) \to h(0)$. Thus there is no way to define $h(0)$ so that h becomes continuous at $x = 0$. $\square$

Problem 21 E. Let g be any linear function from $\mathbf{R}^2$ to $\mathbf{R}^3$. Show that not every element of $\mathbf{R}^3$ is the image under g of a vector in $\mathbf{R}^2$.

Solution

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the two standard basis vectors in $\mathbf{R}^2$. Then $g(e_1)$ and $g(e_2)$ are two particular vectors in $\mathbf{R}^3$. Every vector $g(x)$ in the range of g is a linear combination of these two vectors, because if $x = (a, b) = ae_1 + be_2$ then $g(x) = ag(e_1) + bg(e_2)$. All of these vectors lie on the plane determined by $g(e_1)$ and $g(e_2)$ (or possibly on a single line if $g(e_1)$ and $g(e_2)$ lie on one line through the origin). So, we believe that the range of g can be at most a plane within $\mathbf{R}^3$, and cannot be the entire space.

To prove this precisely, let z be any nonzero vector in $\mathbf{R}^3$ that is perpendicular to both $g(e_1)$ and $g(e_2)$ (there are many ways to show that such a vector exists, for example by using cross products). If z was in the range of g then we would have $z = g(x) = ag(e_1) + bg(e_2)$ for some vector $x = (a, b) \in \mathbf{R}^2$. But z is perpendicular to both $g(e_1)$ and $g(e_2)$, so it is perpendicular to any linear combination of them, and hence is perpendicular to itself:

Since D_1, D_2 are both nonempty, we have $f(D) = \{0, 1\}$. Hence the only thing that we have to show is that f is continuous. There are many ways to do this; I will do it by using the Global Continuity Theorem (Theorem 22.1).

Let G be any open set in $\mathbf{R}$. We must show that there exists an open set $G_1 \in \mathbf{R}^p$ such that $f^{-1}(G) = G_1 \cap D$. There are several cases to consider.

If G contains neither 0 nor 1 then $f^{-1}(G) = \emptyset = \emptyset \cap D$. The set $G_1 = \emptyset$ is open, so this case is done.

If G contains both 0 and 1 then $f^{-1}(G) = D = \mathbf{R}^p \cap D$, and $G_1 = \mathbf{R}^p$ is open.

If G contains 0 but not 1 then $f^{-1}(G) = D_1 = A \cap D$, and $G_1 = A$ is open by definition of disconnection.

Finally, if G contains 1 but not 0 then $f^{-1}(G) = D_2 = B \cap D$, and $G_1 = B$ is open.

"$\Leftarrow$" Suppose that there exists a continuous function $f \colon D \to \mathbf{R}$ such that $f(D) = \{0, 1\}$. The interval $I_1 = (-1/2, 1/2)$ is open in $\mathbf{R}$, so by the Global Continuity Theorem, there exists an open set $A \subseteq \mathbf{R}^p$ such that $f^{-1}(I_1) = A \cap D$. Similarly, $I_2 = (1/2, 3/2)$ is open, so there exists an open set $B \subseteq \mathbf{R}^p$ such that $f^{-1}(I_2) = B \cap D$. I will show that (A, B) is a disconnection of D.

First, $R(f) = f(D) = \{0, 1\}$ and $0 \in I_1$, so $A \cap D = f^{-1}(I_1) \neq \emptyset$. Similarly, $1 \in I_2$, so $B \cap D = f^{-1}(I_2) \neq \emptyset$.

Second,

$$(A \cap D) \cap (B \cap D) = f^{-1}(I_1) \cap f^{-1}(I_2) = f^{-1}(I_1 \cap I_2) = f^{-1}(\emptyset) = \emptyset.$$

Finally, since $R(f) \subseteq I_1 \cup I_2$, we have

$$(A \cap D) \cup (B \cap D) = f^{-1}(I_1) \cup f^{-1}(I_2) = f^{-1}(I_1 \cup I_2) = D.$$

Since A, B are both open, we conclude that (A, B) is indeed a disconnection and therefore D is disconnected. $\square$

Problem 23 L. If $f \colon [0, 1] \to [0, 1]$ is continuous, show that f has a fixed point in $[0, 1]$. Hint: Consider $g(x) = f(x) - x$.

Solution

We must show that there is an $x \in [0, 1]$ such that $g(x) = 0$.

If $g(0) = 0$ or $g(1) = 0$ then we are done. So, suppose that $g(0) \neq 0$ and $g(1) \neq 0$. Then $g(0) = f(0) - 0 = f(0) > 0$ and $g(1) = f(1) - 1 < 0$ since $f(0)$ and $f(1)$ are both between 0 and 1. Hence g is strictly positive at $x = 0$ and strictly negative at $x = 1$. Since g is continuous, it therefore follows from the Intermediate Value Theorem that there must be some x such that $g(x) = 0$. $\square$