MATH 4317 (C. Heil) EXAM #2 July 12, 2007

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (15 points) and choose TWO other problems to answer (10 points each). You may also answer (for up to 3 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem. There are problems on BOTH SIDES of this page!

All statements require proof or justification. There are 35 points total, plus up to 3 points of extra credit.

1. Prove the following statements. Note that the parts of this problem are not related to each other.

a. Let K be a compact subset of $\mathbf{R}^p$, and let $r > 0$ be fixed. Prove directly from the definition of compact set that there exist finitely many points $x_1, \dots, x_N \in K$ such that

$$\forall\, y \in K, \quad \exists\, n \in \{1, \dots, N\} \text{ such that } \|y - x_n\| < r.$$

Hint: Consider balls $B_r(x)$ with $x \in K$.

b. For each $n \in \mathbf{N}$, define a function $f_n \colon \mathbf{R} \to \mathbf{R}$ by

$$f_n(x) \;=\; \begin{cases} 1, & n < x < n+1, \\ 0, & x \le n \text{ or } x \ge n+1. \end{cases}$$

Show that f_n converges pointwise to the zero function on $\mathbf{R}$, but that f_n does not converge uniformly to the zero function.

c. Let

$$x_n \;=\; \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \cdots + \frac{1}{n \cdot 2^n}.$$

Prove directly from the definition that $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence in $\mathbf{R}$.

Remark: You can use without proof, if you like, the fact that $\sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^m}$.

2. Let $S = \{s_n : n \in \mathbf{N}\}$ be a set of strictly positive real numbers such that $\inf(S) = 0$. Let $A \subset \mathbf{R}^p$. Suppose that $x \in \mathbf{R}^p$ is such that for each $n \in \mathbf{N}$ there exists a point $y_n \in A \setminus \{x\}$ such that $\|x - y_n\| < s_n$. Prove directly from the definition of cluster point that x is a cluster point of A.

Note: The definition of cluster point is as follows: We say that x is a cluster point of A if for every neighborhood N of x there exists a point $y \in A \cap N$ with $y \neq x$.

3. Let $(x_n)_{n \in \mathbf{N}}$ be a sequence of real numbers with $x_n > 0$ for every n. Show that if $\lim\,(x_n^{1/n}) > 1$, then $(x_n)_{n \in \mathbf{N}}$ is not convergent.

Hint: Show that it is not a bounded sequence.

4. Suppose that for each $n \in \mathbf{N}$ we are given a piecewise continuous function $f_n \colon [0,1] \to \mathbf{R}$, and another piecewise continuous function $f \colon [0,1] \to \mathbf{R}$. Show that if $f_n \to f$ uniformly, then

$$\sup_{n \in \mathbf{N}} \|f_n\|_\infty \;<\; \infty.$$

Will it also be true that $\sup \|f_n\|_1 < \infty$?

5. Let A be any subset of $\mathbf{R}^p$. Let A^- be the closure of A and let ∂A be the boundary of A. Prove that

$$A^- \;=\; A \cup \partial A.$$

Hint: Prove that $A \cup \partial A$ is closed.