MATH 4317 (C. Heil) EXAM #1 June 12, 2007

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (15 points) and choose TWO other problems to answer (10 points each). You may also answer (for up to 3 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem.

All statements require proof or justification. There are 35 points total, plus up to 3 points of extra credit.

1. Prove the following statements. Note: Part a is not related to parts b or c of this problem.

a. Show that $\inf\left\{\frac{n+2}{n+1} : n \in \mathbf{N}\right\} = 1.$

Hint: First show that 1 is a lower bound, and then show that it is the greatest lower bound.

b. Let A and B be subsets of $[0,\infty)$ that are bounded above. Define the set AB to be

$$AB = \{ab : a \in A,\, b \in B\}.$$

Prove that AB is bounded above, and that

$$\sup(AB) \le \sup(A)\sup(B).$$

c. In part b we required A and B to only contain nonnegative numbers. For this part, let A and B be any bounded subsets of $\mathbf{R}$. Either prove that

$$\sup(AB) = \sup(A)\sup(B),$$

or find an example of sets A and B for which equality does not hold.

2. a. Let A and B be two denumerable sets, such that $A \cap B = \emptyset$. Show that $A \cup B$ is denumerable by exhibiting a $1-1$ correspondence between $A \cup B$ and $\mathbf{N}$.

b. Let X be an uncountable set, and let Y be a denumerable subset of X. Show that $X \setminus Y$ is uncountable. Hint: use part a.

c. Prove that $\{x \in \mathbf{R} : x \text{ is irrational}\}$ is uncountable.

3. a. Let X, Y, Z be sets, and suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are functions such that $g \circ f\colon X \to Z$ is a bijection. Show that f is injective and g is surjective.

b. Give an example of sets X, Y, Z and functions $f\colon X \to Y$ and $g\colon Y \to Z$ such that $g \circ f\colon X \to Z$ is a bijection but f is not a surjection and g is not an injection.

4. Let $A \subseteq \mathbf{R}^p$ be given, and define the *interior* of A to be the set

$$A^\circ = \{x \in \mathbf{R}^p : x \text{ is an interior point of } A\}.$$

a. Prove that if U is an open set and $U \subseteq A$, then $U \subseteq A^\circ$.

b. Prove that if A, B are two subsets of $\mathbf{R}^p$, then $(A \cap B)^\circ = A^\circ \cap B^\circ$.