Here are a few practice problems on groups. The first ones are easier and the later ones are harder.

1. Let $A(\mathbb{R})$ be the set of all bijections of the real line $\mathbb{R}$ onto itself. We know that this is a group under the operation of composition of functions (you do not need to prove that fact).

   (a) Identify three different functions that belong to $A(\mathbb{R})$, i.e., give a formula for three functions that belong to $A(\mathbb{R})$.
   
   Note: You don’t need to prove that your functions are bijections, you just need to give formulas for three functions that are bijections.

   (b) Identify the identity element of $A(\mathbb{R})$, i.e., give the formula for the function that is the identity of $A(\mathbb{R})$. Prove that your function is indeed the identity element of $A(\mathbb{R})$.

   (c) Let $f(x) = x + 1$. Determine (with proof) whether $f$ has finite order or not.

2. Let $G$ and $H$ be groups, and let $f: G \to H$ be a homomorphism.

   (a) Suppose that $x \in G$. Let $k$ be the order of $x$, and let $m$ be the order of $f(x)$. Without appealing to any theorems on order, show directly that $m$ divides $k$.
   
   Hint: $k = mj + r$.

   (b) Show that if $|G|$ and $|H|$ are relatively prime (no common divisors) then $\ker(f) = G$.

3. Suppose that $M$ is a normal subgroup of a group $G$, and $N$ is a normal subgroup of a group $H$. Then the set $M \times N$ is a normal subgroup of $G \times H$ (you do not need to prove this).

   (a) Define $f: G \times H \to (G/M) \times (H/N)$ by
   
   $$f((a, b)) = (Ma, Nb), \quad (a, b) \in G \times H.$$ 
   
   Show that $f$ is a surjective homomorphism.

   (b) Use the First Homomorphism Theorem to show that
   
   $$(G \times H)/(M \times N) \cong (G/M) \times (H/N).$$

4. Suppose that $G$ is a finite abelian group with order $|G| > 1$. Suppose there exists a prime number $p$ such that for each $a \in G$ there exists a positive integer $k$ such that $a^{pk} = e$ (the integer $k$ depends on the element $a$). Show that $|G| = p^n$ for some integer $n$. 

5. Suppose that $M, N$ are normal subgroups of a group $G$, and $M \cap N = \{e\}$. Prove that $MN$ is a normal subgroup of $G$, and $M \times N \cong MN$.

6. Suppose that $M, N$ are normal subgroups of a group $G$, and $MN = G$.
   
   (a) Define $f : G \to G/M \times G/N$ by $f(a) = (Ma, Na)$ for $a \in G$. Prove that $f$ is a surjective homomorphism of $G$ onto $G/M \times G/N$.
   
   Hint: Surjective is the hard part, do all the other parts of this problem first.
   
   (b) Prove that $\ker(f) = M \cap N$.
   
   (c) Given the results from parts (a) and (b), what does the First Homomorphism Theorem now imply?

7. Let $G$ be a group. Let $C$ be the set of all commutators of elements of $G$, i.e.,
   $$ C = \{xyx^{-1}y^{-1} : x, y \in G\}. $$

   Unfortunately, $C$ need not be a subgroup of $G$. Therefore we define the commutator subgroup $C'$ to be the subgroup "generated by" $C$. Specifically, this means that $C'$ is the intersection of all subgroups of $G$ that contain $C$:
   $$ C' = \bigcap\{H : H \text{ is a subgroup of } G \text{ and } C \subseteq H\}. $$

   (a) Prove that $C'$ is a normal subgroup of $G$ (prove both that $C'$ is a subgroup, and that it is normal).
   
   (b) Prove that $G/C'$ is abelian.