MATH 4107 (Prof. Heil) PRACTICE PROBLEMS WITH SOLUTIONS Spring 2018

Here are a few practice problems on groups. **You should first work through these WITHOUT LOOKING at the solutions!** After you write your own solution, you can compare to my solution. Your solution does not need to be identical to mine—there are often many ways to solve a problem—but it does need to be CORRECT.

1. Suppose that G, A, and B are groups and $\psi_1 \colon G \to A$ and $\psi_2 \colon G \to B$ are surjective homomorphisms. Suppose further that $\ker(\psi_1) \cap \ker(\psi_2) = \{e\}$.

(a) Define $\psi \colon G \to A \times B$ by

$$\psi(g) \;=\; (\psi_1(g), \psi_2(g)), \qquad g \in G,$$

is an injective homomorphism of G into $A \times B$.

Solution

If g, $h \in G$, then, since ψ_1, ψ_2 are both homomorphisms,

$$\begin{aligned} \psi(gh) &= \big(\psi_1(gh), \psi_2(gh)\big) \\ &= \big(\psi_1(g)\psi_1(h),\, \psi_2(g)\psi_2(h)\big) \\ &= \big(\psi_1(g), \psi_2(g)\big)\;\big(\psi_1(h), \psi_2(h)\big) \\ &= \psi(g)\psi(h). \end{aligned}$$

Thus ψ is a homomorphism.

To show that ψ is injective, we compute its kernel. Suppose that $\psi(g) = e_{A\times B}$. Then since $\psi(g) = (\psi_1(g), \psi_2(g))$ and $e_{A\times B} = (e_A, e_B)$, we have $\psi_1(g) = e_A$ and $\psi_2(g) = e_B$. Hence $g \in \ker(\psi_1) \cap \ker(\psi_2) = \{e\}$, where $e = e_G$ is the identity of G. Therefore $\ker(\psi) = \{e\}$, so ψ is injective. □

(b) Show by example that ψ does not always need to be surjective.

Solution

Consider the case where $A = B$ are the same group, and $G = \{(a,a) : a \in A\}$. Define $\psi_i \colon G \to A$ by $\psi_i(a,a) = a$. Then for $g = (a,a) \in G$, we have

$$\psi(g) \;=\; \psi(a,a) \;=\; (\psi_1(a,a),\, \psi_2(a,a)) \;=\; (a,a) \;=\; g.$$

Thus ψ is just the inclusion mapping of G into $A \times B$. However, ψ cannot be injective unless A contains only a single element. For, if A contained two distinct elements $a \neq b$, then $(a,b) \in A \times B$ yet $(a,b) \notin G = \text{range}(\psi)$. □

2. Classify the finite abelian groups of order 108.

Solution
Since $108 = 4 \cdot 27 = 2^2 \cdot 3^3$, every finite abelian group of order 108 is isomorphic to one of the following groups:

$$\begin{aligned}
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_3, \\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27}, \\
&\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \\
&\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_3, \\
&\mathbb{Z}_4 \times \mathbb{Z}_{27}. \quad \square
\end{aligned}$$

3. Prove that $\mathbb{Z}_9 \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Solution
The group $\mathbb{Z}_9$ contains an element of order 9, namely 1. However, $\mathbb{Z}_3 \times \mathbb{Z}_3$ does not. To see this, suppose that we take any ordered pair $(a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $a \in \mathbb{Z}_3$, so $a+a+a = 3a = 0$ in $\mathbb{Z}_3$. The same is true for b, so

$$(a, b) + (a, b) + (a, b) \;=\; (a + a + a, b + b + b) \;=\; (0, 0).$$

Hence every element of $\mathbb{Z}_3 \times \mathbb{Z}_3$ has order *at most* 3. By homework problem 2.58(ii), these two groups cannot be isomorphic. $\square$