MATH 4107 (Prof. Heil) PRACTICE PROBLEMS WITH SOLUTIONS Spring 2018

Here are a few practice problems on groups. **You should first work through these WITHOUT LOOKING at the solutions!** After you write your own solution, you can compare to my solution. Your solution does not need to be identical to mine—there are often many ways to solve a problem—but it does need to be CORRECT.

1. Suppose that N is a normal subgroup of a group G. Show that if $a \in G$ has finite order $o(a)$, then Na in G/N has finite order m with $m|o(a)$.

Solution

Let $n = o(a)$. Then $a^n = e$, so

$$(Na)^n = Na^n = Ne = N,$$

which is the identity element of G/N. Therefore Na has finite order, and its order must divide n. In other words, we must have $o(Na)|n$. □

Question: Must it be true that $o(Na) = o(a)$? (No. Try an example!)

2. Suppose that H and K are groups. I won't prove it here, but you should be able to prove that the Cartesian product

$$H \times K = \{(h,k) : h \in H,\, k \in K\}$$

is a group with group operation $(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)$. Prove the following facts about this group.

(a) Show that $H^* = \{(h, e_K) : h \in H\}$ and $K^* = \{(e_H, k) : k \in k\}$ are normal subgroups of $H \times K$.

Solution

The fact that H^* and K^* are subgroups of $H \times K$ is a straightforward verification of the definition of subgroup, and I will leave this part to you. To see that H^* is a normal subgroup, suppose that $(h, e_K) \in H^*$ and (x, y) is any element of $H \times K$. Then:

$$(x,y)\,(h,e_K)\,(x,y)^{-1} = (x,y)\,(h,e_K)(x^{-1},y^{-1}) = (xhx^{-1}, ye_Ky^{-1}) = (xhx^{-1}, e_K) \in H^*.$$

This shows that $(x,y)H^*(x,y)^{-1} \subseteq H^*$, so we conclude that H^* is normal. A completely symmetric proof then shows that K^* is normal as well. □

(b) Show that $H \cong H^*$ and $K \cong K^*$.

Solution

To see that H and H^* are isomorphic, define a mapping $f\colon H \to H^*$ by

$$f(h) = (h, e_K), \qquad h \in H.$$

This is a homomorphism because given any $a,\, b \in H$ we have

$$f(ab) \;=\; (ab, e_K) \;=\; (a, e_K)\,(b, e_K) \;=\; f(a)f(b).$$

To complete the proof, you must verify that f is a bijection (do it!). Then you have shown that f is an isomorphism, so you conclude that $H \cong H^*$. A similar argument shows that $K \cong K^*$. $\square$

(c) Prove that $(H \times K)/K^* \cong H$. Hint: Find a function that maps $H \times K$ to H, and then use the First Homomorphism Theorem.

<u>Solution</u>

Define $f : H \times K \to H$ by the rule

$$f(h,k) \;=\; h, \qquad (h,k) \in H \times K.$$

This is a homomorphism because

$$f\big((h_1,k_1)\,(h_2,k_2)\big) \;=\; f(h_1h_2, k_1k_2) \;=\; h_1h_2 \;=\; f(h_1,k_1)\,f(h_2,k_2).$$

It is certainly surjective because if $h \in H$ then $f(h, e_K) = h$. We claim that the kernel of f is K^*. First, if $(e_H, k) \in K^*$ then $f(e_H, k) = e_H$, so $(e_H, k) \in \ker(f)$. Second, if $(h,k) \in \ker(f)$ then $h = f(h,k) = e_H$, so $(h,k) = (e_H, k) \in K^*$. Thus $\ker(f) = K^*$.

Thus, we have a surjective homomorphism f that maps $H \times K$ onto H and has kernel K^*. The First Homomorphism Theorem therefore tells us that $H \cong (H \times K)/K^*$. $\square$