Here are a few practice problems on groups. **You should first work through these WITHOUT LOOKING at the solutions!** After you write your own solution, you can compare to my solution. Your solution does not need to be identical to mine—there are often many ways to solve a problem—but it does need to be **CORRECT**.

1. Suppose that $N$ is a normal subgroup of a group $G$. Show that if $a \in G$ has finite order $o(a)$, then $Na$ in $G/N$ has finite order $m$ with $m | o(a)$.

**Solution**

Let $n = o(a)$. Then $a^n = e$, so

$$(Na)^n = Na^n = Ne = N,$$

which is the identity element of $G/N$. Therefore $Na$ has finite order, and its order must divide $n$. In other words, we must have $o(Na) | n$. □

**Question:** Must it be true that $o(Na) = o(a)$? (No. Try an example!)

2. Suppose that $H$ and $K$ are groups. I won’t prove it here, but you should be able to prove that the Cartesian product $H \times K = \{(h, k) : h \in H, k \in K\}$ is a group with group operation $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$. Prove the following facts about this group.

(a) Show that $H^* = \{(h, e_K) : h \in H\}$ and $K^* = \{(e_H, k) : k \in K\}$ are normal subgroups of $H \times K$.

**Solution**

The fact that $H^*$ and $K^*$ are subgroups of $H \times K$ is a straightforward verification of the definition of subgroup, and I will leave this part to you. To see that $H^*$ is a normal subgroup, suppose that $(h, e_K) \in H^*$ and $(x, y)$ is any element of $H \times K$. Then:

$$(x, y)(h, e_K)(x, y)^{-1} = (x, y)(h, e_K)(x^{-1}, y^{-1}) = (xhx^{-1}, ye_Ky^{-1}) = (xhx^{-1}, e_K) \in H^*.$$ 

This shows that $(x, y)H^*(x, y)^{-1} \subseteq H^*$, so we conclude that $H^*$ is normal. A completely symmetric proof then shows that $K^*$ is normal as well. □

(b) Show that $H \cong H^*$ and $K \cong K^*$.

**Solution**

To see that $H$ and $H^*$ are isomorphic, define a mapping $f : H \to H^*$ by

$$f(h) = (h, e_K), \quad h \in H.$$
This is a homomorphism because given any $a, b \in H$ we have
$$f(ab) = (ab, e_K) = (a, e_K)(b, e_K) = f(a)f(b).$$
To complete the proof, you must verify that $f$ is a bijection (do it!). Then you have shown that $f$ is an isomorphism, so you conclude that $H \cong H^*$. A similar argument shows that $K \cong K^*$.

(c) Prove that $(H \times K)/K^* \cong H$. Hint: Find a function that maps $H \times K$ to $H$, and then use the First Homomorphism Theorem.

Solution
Define $f : H \times K \to H$ by the rule
$$f(h, k) = h, \quad (h, k) \in H \times K.$$ This is a homomorphism because
$$f((h_1, k_1)(h_2, k_2)) = f(h_1h_2, k_1k_2) = h_1h_2 = f(h_1, k_1)f(h_2, k_2).$$
It is certainly surjective because if $h \in H$ then $f(h, e_K) = h$. We claim that the kernel of $f$ is $K^*$. First, if $(e_H, k) \in K^*$ then $f(e_H, k) = e_H$, so $(e_H, k) \in \ker(f)$. Second, if $(h, k) \in \ker(f)$ then $h = f(h, k) = e_H$, so $(h, k) = (e_H, k) \in K^*$. Thus $\ker(f) = K^*$.

Thus, we have a surjective homomorphism $f$ that maps $H \times K$ onto $H$ and has kernel $K^*$. The First Homomorphism Theorem therefore tells us that $H \cong (H \times K)/K^*$.