Aside: Functions whose domain is a set of functions.

Let

$$\mathcal{F} = \{ f : f \text{ is a function from } \mathbb{R} \text{ to } \mathbb{R} \}.$$

The set $\mathcal{F}$ [redacted] consists of all possible functions that map real numbers to real numbers. So for example, if $f(x) = e^x$ then $f \in \mathcal{F}$.

Notation: Do NOT write $f(x) \in \mathcal{F}$.

$f(x)$ denotes the value of f at x, and [redacted] is a real number. For example, if $f(x) = e^x$ then $f(1) = e^1 = e$. The number e is not a function: $f(1) = e \notin \mathcal{F}$. The function f whose rule is $f(x) = e^x$ does belong to $\mathcal{F}$: $f \in \mathcal{F}$.

Sometimes (often) we abuse notation and write $e^x \in \mathcal{F}$, but you must realize that this is incorrect notation, used only because we are lazy.

Now I want to consider the group

$$A(\mathcal{F}) = \{ F : \mathcal{F} \to \mathcal{F} \ : \ F \text{ is a bijection} \}.$$

To be an element of $A(\mathcal{F})$, we must have a function $F : \mathcal{F} \to \mathcal{F}$ that is a bijection. So, F inputs an element of $\mathcal{F}$ (a function on $\mathbb{R}$) and outputs a new function $F(f)$. Since $F(f)$ is a new function mapping real numbers to real numbers, $F(f)$ is a rule that maps a <u>number x</u> to a number $F(f)(x)$. Since the multitude of parentheses is often confusing we sometimes write

$$Ff \quad \text{instead of } F(f).$$

So Ff is a function, & it takes values $(Ff)(x)$ for $x \in \mathbb{R}$.

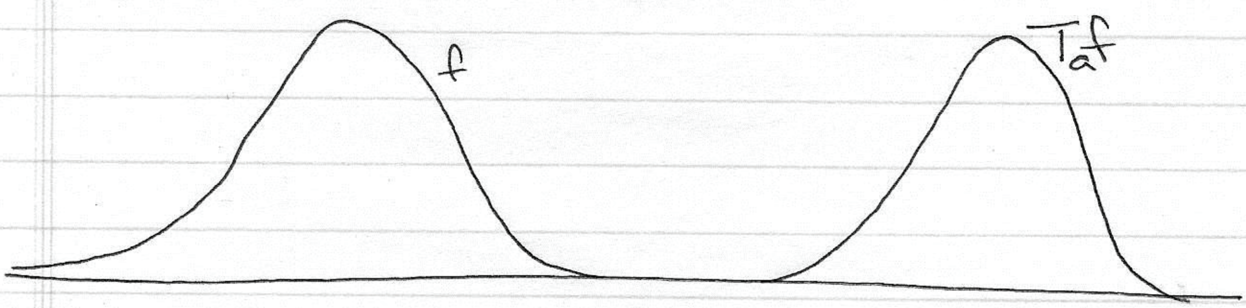
Now let us consider some specific examples of functions $F \in A(\mathcal{F})$.

<u>Translation</u>

Given a fixed real number $a \in \mathbb{R}$, we define a function $T_a \in A(\mathcal{F})$ as follows. Given $f \in \mathcal{F}$ we want $T_a f$ to be a new function on $\mathbb{R}$. We define $T_a f$ to be the function whose rule is

$$(T_a f)(x) = f(x-a).$$

The graph of the function $T_a f$ is the graph of f shifted a units to the right.



For example, if $a = \pi/2$ and $f(x) = \sin x$, then $T_{\frac{\pi}{2}} f$ is the function whose rule is

$$\left(T_{\frac{\pi}{2}} f\right)(x) = f\left(x - \frac{\pi}{2}\right) = \sin\left(x - \frac{\pi}{2}\right) = \cos x.$$

We have that $T_a : \mathcal{F} \to \mathcal{F}$, i.e., T_a maps functions to functions.

<u>Exercise</u>: Show that T_a is a bijection of $\mathcal{F}$ onto itself.

This does <u>not</u> mean that $T_a f$ is a bijection for each $f \in \mathcal{F}$. Rather what you have to show is that:

T_a is 1-1: if $T_a f = T_a g$ then $f = g$,

T_a is onto: if $g \in \mathcal{F}$ then $\exists f \in \mathcal{F}$ s.t. $T_a f = g$.

Each different $a \in \mathbb{R}$ gives us a different $T_a \in A(\mathcal{F})$. However, these are not the <u>only</u> elements of $A(\mathcal{F})$. Let us give another example

Dilation

Suppose that $a > 0$ is a fixed real number.

Define a function $D_a : \mathcal{F} \to \mathcal{F}$ as follows.

Given $f \in \mathcal{F}$, we define $D_a f$ to be the function

whose rule is

$$(D_a f)(x) = f(ax), \qquad x \in \mathbb{R}.$$

The functions $f \in \mathcal{F}$ need not be bijections of $\mathbb{R}$

onto $\mathbb{R}$. However, the mapping from f to

$D_a f$ is a bijection of $\mathcal{F}$ onto $\mathcal{F}$.

Exercise: Given a fixed $a \in \mathbb{R}$, show that

$D_a \in A(\mathcal{F})$, ie., D_a is a bijection of

$\mathcal{F}$ onto $\mathcal{F}$.

Example

Suppose $a = 2$ and $f(x) = e^x$.

Then $D_a f$ is the function whose rule is

$$(D_2 f)(x) = f(2x) = e^{2x}.$$

<u>Compositions</u>

Given [redacted] $a > 0$ & $b \in \mathbb{R}$, we have that $D_a, T_b \in A(\mathcal{F})$. Since $A(\mathcal{F})$ is a group under compositions, we know that $D_aT_b = D_a \circ T_b$ belongs to $A(\mathcal{F})$. What is the composition D_aT_b? It is a bijection of $\mathcal{F}$ onto $\mathcal{F}$ given by

$$(D_aT_b)(f) = D_a(T_b(f)).$$

Now, each of these are <u>functions</u> that map real numbers to real numbers. f is the <u>input</u> to D_aT_b, f is <u>not</u> being composed with anything. Rather, the dilation D_a is being composed with the translation T_b, & the result is evaluated at the function f. Each of

$$(D_a T_b)(f) \quad \text{and} \quad D_a(T_b(f))$$

are functions on $\mathbb{R}$. To say that they are equal functions means that they act the same on each number $x \in \mathbb{R}$, i.e.,

$$\Big((D_a T_b)(f)\Big)(x) = \Big(D_a(T_b(f))\Big)(x).$$

The right-hand side says that the function f is translated by b, then dilated by a, & the result is evaluated at x. Let us work out what the result is. For simplicity of notation, let's cut down on the parentheses & write $T_b f$ for $T_b(f)$. Then:

$$\big((D_a T_b)(f)\big)(x) = \big(D_a(T_b f)\big)(x)$$

$$= (D_a g)(x) \quad \text{where } g = T_b f$$

$$= g(ax)$$

$$= (T_b f)(ax)$$

$$= (T_b f)(y) \quad \text{where } y = ax$$

$$= f(y-b)$$

$$= f(ax-b).$$

<u>Exercise</u>

Show that

$$\big((T_b D_a)(f)\big)(x) = f(ax-ab)$$

Notation

Again for simplicity of notation we often omit some parentheses, and write

$$D_a T_b f \quad \text{for} \quad (D_a T_b)(f) = D_a(T_b(f)).$$

Thus we have shown that

$$(D_a T_b f)(x) = f(ax-b).$$

Important note

We CANNOT write

$$(D_a T_b f)(x) = D_a(T_b(f(x))) \quad !! \qquad (*)$$

First, this doesn't make any sense – $f(x)$ is a scalar, not a function, so $T_b(f(x))$ doesn't make sense. You can't write $T_b(\pi)$ – the scalar π is not in the domain of T_b. T_b is a function whose inputs are functions, not numbers. $(T_b(f))(x)$ makes sense, & we usually abbreviate it as $(T_b f)(x)$, but $T_b(f(x))$ does NOT make sense.

Second, when you try to write the incorrect formula (*) you are trying to think of it as

$$(D_a \circ T_b \circ f)(x).$$

However, f is not being composed with anything! D_a & T_b are functions whose domain is $\mathcal{F}$ and f belongs to $\mathcal{F}$. f is the <u>input</u> to the composition $D_a \circ T_b$, it is not being <u>composed</u> with $D_a \circ T_b$. With all the parentheses written out, $(D_a T_b f)(x)$ means

$$\Big((D_a \circ T_b)(f) \Big)(x)$$

$$= \big(D_a(T_b(f))\big)(x).$$

Exercises

a. Given $a \in \mathbb{R}$, we have that T_a belongs to the group $A(\mathcal{F})$. What is the inverse of T_a in this group? Show that $T_a^{-1} = T_{-a}$.

b. Given $a > 0$, show that $D_a^{-1} = D_{1/a}$.

c. Given $a > 0$ & $b \in \mathbb{R}$, show that

$$T_b D_a = D_a T_{ab}.$$

To do this, you must show that

$$(T_b D_a)(f) = (D_a T_{ab})(f) \quad \forall f \in \mathcal{F}.$$

And to do <u>this</u>, you must show that

$$\big((T_b D_a)(f)\big)(x) = \big((D_a T_{ab})(f)\big)(x) \quad \forall f \in \mathcal{F}$$
$$\forall x \in \mathbb{R}.$$

Usually we write this in shorthand notation as

$$(T_b D_a f)(x) = (D_a T_{ab} f)(x) \quad \forall f \in \mathcal{F},\ \forall x \in \mathbb{R}.$$

Exercises

a. Show that $H = \{T_a\}_{a \in \mathbb{R}}$ is a subgroup of $A(\mathcal{F})$.

b. Let $\mathbb{R}$ be the group of reals under addition. Define

$$U : \mathbb{R} \longrightarrow H$$
$$U(a) = T_a$$

Show that U is an isomorphism, i.e., $H \cong \mathbb{R}$.

Exercises

a. Show that $K = \{D_a\}_{a>0}$ is a subgroup of $A(\mathcal{F})$.

b. Let $\mathbb{R}^+ = (0, \infty)$ be the positive reals under multiplication. Define

$$V : \mathbb{R}^+ \longrightarrow K$$
$$V(a) = D_a$$

Show that V is an isomorphism, so $K \cong \mathbb{R}^+$.

Exercises

a. Show that

$$A = \{ D_a T_b : a > 0, b \in \mathbb{R} \}$$

is a subgroup of $A(\mathcal{F})$.

b. Let G be the affine group (see chapter 1).
Show that $G \cong A$.