4.4 Maximal Ideals

Definition

A ring R is *simple* if it has no nontrivial ideals, i.e., $\{0\}$ & R are the only ideals.

Recall that a commutative ring is simple if & only if it is a field.

Definition

An ideal M in a ring R is a *maximal ideal* if

a. $M \subsetneq R$ (meaning M is a proper subset of R)

b. There are no ideals I with $M \subsetneq I \subsetneq R$.

(so there are no ideals strictly between M and R)

Note: Thus

R is a simple ring $\iff$ $\{0\}$ is a maximal ideal.

Exercise: $\mathbb{Z} + \sqrt{2}\,\mathbb{Z} = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ is not a simple ring (hence is not a field)

Theorem

Let M be a proper ideal in a ring R. Then:

$$M \text{ is maximal} \iff R/M \text{ is simple.}$$

Proof:

$\Leftarrow$ Suppose that R/M is simple. Suppose that I is an ideal in R & $M \subseteq I \subseteq R$.

Must show that either $I = M$ or $I = R$

Consider the canonical map

$$\pi : R \longrightarrow R/M$$
$$\pi(a) = a + M.$$

Consider

I is an ideal in R, and J is the set of corresponding cosets of M using elements of I

$$J = \pi(I) = \{a + M : a \in I\}.$$

Exercise: Show J is an ideal in R/M.

But R/M (by hypothesis) is simple, so either $J = \{M\}$ (since $M = 0_{R/M}$)

(since M is the zero element of R/M), or

$J = R/M$.

Alternative explanation:
$J = \{a+M : a \in I\}$, so if
$J = \{M\}$ then $a+M = M$ $\forall a \in I$
But $a+M=M \iff a \in M$. So $I \subseteq M$

Case 1: $J = \{M\}$.

We claim that this implies that $I = M$.

To see this, suppose that $a \in I$. Then, by definition of J, we have $a+M \in J$. But $J = \{M\}$, so this implies $a+M = M$. Hence $a \in M$, so we have shown that $I \subseteq M$.

By hypothesis we have $M \subseteq I$.

Thus, in this case we have $I = M$.

Case 2: $J = R/M$.

Recall $J = \{a+M : a \in I\}$

We claim that this implies that $I = R$.

Note that we have $I \subseteq R$, so we only have to show the reverse inclusion. Suppose that $a \in R$. Then $a+M \in R/M = J$, so $a+M = b+M$ for some $b \in I$. But then $a-b \in M$, and

$M \subseteq I$ so $a - b \in I$. Since $b \in I$, we
by hypothesis

conclude $a = (a-b) + b \in I$. Therefore $R = I$.

Thus there are only two possibilities for I: either $I = M$ or $I = R$. Therefore M is maximal.

$\Rightarrow$ Exercise: Show that if M is maximal then R/M is simple.

Hint: [redacted] Suppose that J is an ideal in R/M. Show that

$$I = \pi^{-1}(J) = \{a \in R : a + M \in J\}$$

is an ideal in R, and $M \subseteq I \subseteq R$. Since M is maximal, this leaves only two possibilities. ∎

Corollary

Let R be a commutative ring with identity, and let M be a proper ideal in R. Then:

$$M \text{ is maximal} \iff R/M \text{ is a field.}$$

Proof:

$\Leftarrow$ Suppose that R/M is a field. Then we know by earlier results that R/M is simple. The preceding theorem therefore implies that M is maximal.

$\Rightarrow$ Suppose that M is maximal. Then the preceding theorem implies that R/M is simple. Thus R/M is a commutative ring with 1 that has no nontrivial ideals. We proved earlier that this implies that R/M is a field. ∎

<u>Example/Exercise</u>

Consider $R = \mathbb{Z}$. Suppose that $n > 0$ is composite, i.e., $n = k\ell$ with $k, \ell > 1$.

Exercise: Show $n\mathbb{Z} \subsetneq k\mathbb{Z} \subsetneq \mathbb{Z}$.

Thus $n\mathbb{Z}$ is not a maximal ideal. Therefore $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ is not a field.

Suppose $p > 0$ is prime. Show that $p\mathbb{Z}$ is a maximal ideal, & conclude that $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ is a field.