2.2 Some Simple Remarks

Theorem

Let G be a group. Then the following statements hold.

a. Cancellation Laws: $xa = xb \Rightarrow$ [redacted] $a = b$

and $ax = bx \Rightarrow a = b$

b. The identity element is unique.

c. Each a in G has a unique inverse $a^{-1} \in G$.

d. $(a^{-1})^{-1} = a$

e. $(ab)^{-1} = b^{-1}a^{-1}$.

Proof:

a. If $xa = xb$, then

$$x^{-1}(xa) = (x^{-1}x)a = ea = a$$

These are equal! [redacted] Therefore a=b.

$$x^{-1}(xb) = (x^{-1}x)b = eb =$$ [redacted] b

The other cancellation law is similar.

b. Suppose e & f were both identity elements. Then

$$\forall a \in G, \qquad \underbrace{ae = a = ea}_{(*)} \qquad \& \qquad \underbrace{af = a = fa}_{(**)}$$

Hence

$$fe = f \text{ by } (*)$$

$$fe = e \text{ by } (**)$$

So $e = f$.

c. Suppose b & c were both inverses of a. Then

$$ba = e = ab \qquad \& \qquad ca = e = ac$$

so

$$b(ac) = be = b$$

$$\shortparallel$$

$$(ba)c = ec = c.$$

d. Since $a^{-1}a = e = aa^{-1}$, the inverse of a^{-1} is a, i.e., $(a^{-1})^{-1} = a$.

5. We want to show that $b^{-1}a^{-1}$ is the inverse of ab. If we multiply them together, we get:

$$
\begin{aligned}
(ab)(b^{-1}a^{-1}) &= a(bb^{-1})a^{-1} \quad \text{(by associativity)} \\
&= aea^{-1} \\
&= aa^{-1} \\
&= e
\end{aligned}
$$

and $(b^{-1}a^{-1})(ab) = e$ similarly. We know that ab has a unique inverse element. There is only one element x that satisfies $(ab)x = e = x(ab)$, and we've just found it. The inverse of ab is $x = b^{-1}a^{-1}$ $\blacksquare$

<u>Definition</u> Exponents

$$a^0 = e$$

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}} \quad \text{for } n \geq 1$$

$$a^{-n} = (a^{-1})^n = \underbrace{a^{-1} a^{-1} \cdots a^{-1}}_{n \text{ times}} \quad \text{for } n \geq 1$$

<u>Theorem</u> Exponents work the way we expect.

1. $(ab)^{-1} = b^{-1} a^{-1}$
2. If a & b <u>commute</u>, then $(ab)^n = a^n b^n$
3. $(a^{-1})^n = (a^n)^{-1}$ for all $n \in \mathbb{Z}$
4. $a^m a^n = a^{m+n}$ for all $m, n \in \mathbb{Z}$
5. $(a^m)^n = a^{mn}$ for all $m, n \in \mathbb{Z}$

<u>Proof</u>: Exercise