# Rigid Motions of the Plane

Most of the functions we study in calculus map real numbers to real numbers, $f: \mathbb{R} \to \mathbb{R}$. These are easily visualized by their graphs. More general functions are harder to visualize, but we can still picture $f$ as taking elements of the domain and sending them to elements of the range. We will look at some functions $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

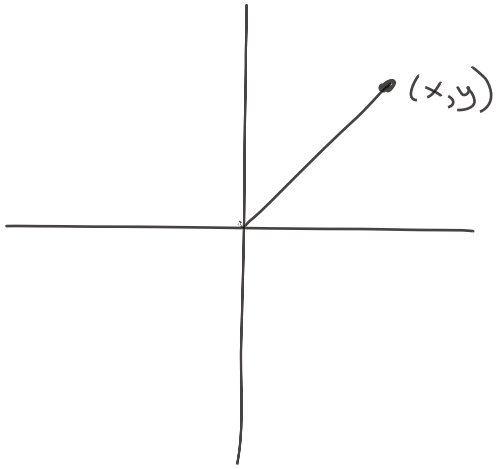
There are many bijections $f: \mathbb{R}^2 \to \mathbb{R}^2$, and some of these are rigid motions

Example: Rotations

For each number $\theta$, define a function $r_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

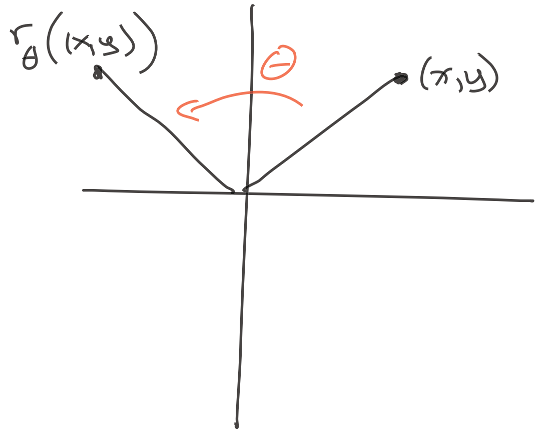
$r_\theta((x,y))$ = the point obtained by rotating $(x,y)$ by an angle of $\theta$ around the origin.

(see picture on next page)




$\mathbb{R}^2$

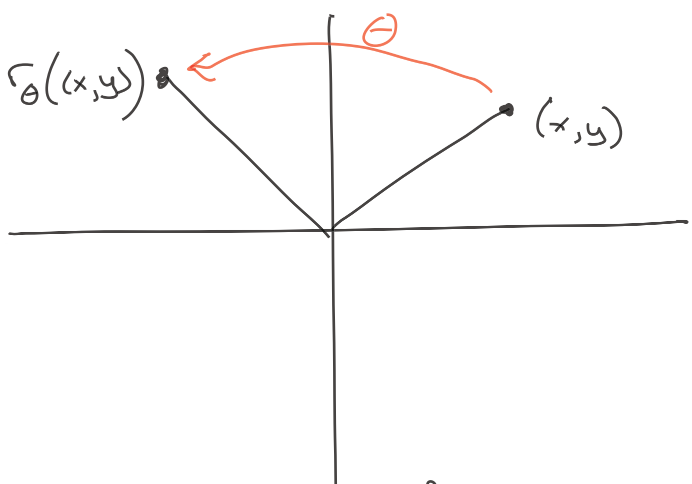
Domain




$\mathbb{R}^2$

Codomain (and range)

We often picture this with the domain & codomain superimposed:




$\mathbb{R}^2$

Note:

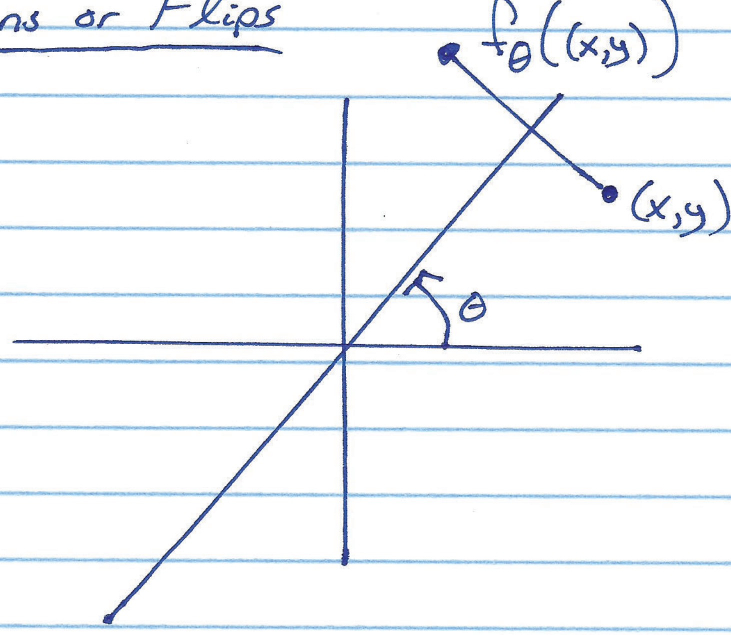
$r_0 = i =$ identity map on $\mathbb{R}^2$ ← We sometimes write this identity map as $i_{\mathbb{R}^2}$, to emphasize that the domain is $\mathbb{R}^2$. Another common choice to denote the identity is $e$

$r_{2\pi} = r_0 = i$

$r_\theta^{-1} = r_{-\theta}$

$r_\theta^2 = r_\theta \circ r_\theta = r_{2\theta}$

## Reflections or Flips




mirror image across the line

$f_\theta$ = reflection across the line at angle $\theta$

Note: $f_\theta^2 = f_\theta \circ f_\theta = id_{\mathbb{R}^2}$

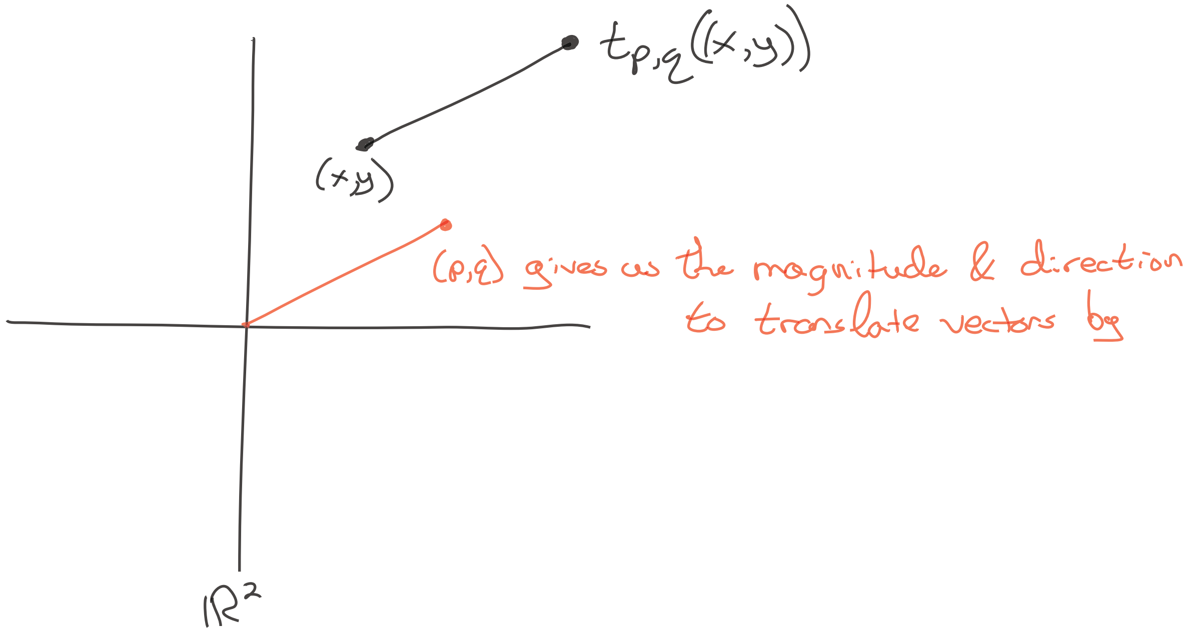
so $f_\theta = f_\theta^{-1}$ !

## Translations

Fix a point $(p,q) \in \mathbb{R}^2$. Then define a rigid motion $t_{p,q} : \mathbb{R}^2 \to \mathbb{R}^2$ that "slides" a point $(x,y)$ to a translated point $(x,y)+(p,q)$:

$$t_{p,q} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$t_{p,q}\big((x,y)\big) = (x,y) + (p,q) = (x+p,\ y+q)$$




Exercise: $t_{p,q}$ is a bijection

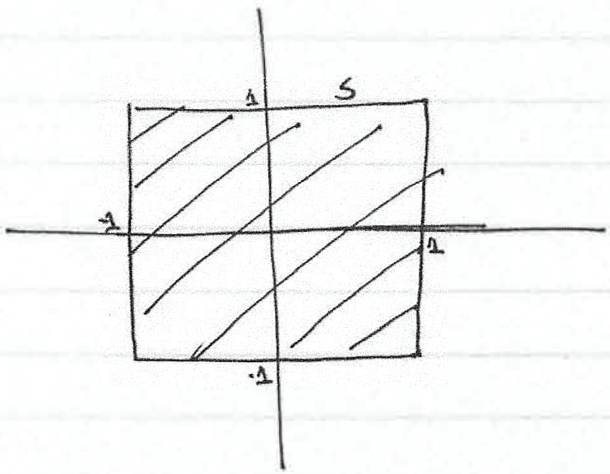
$$t_{p,q}^{-1} = t_{-q,-p} \text{ (translation in the opposite direction)}$$

# Symmetries of the Square



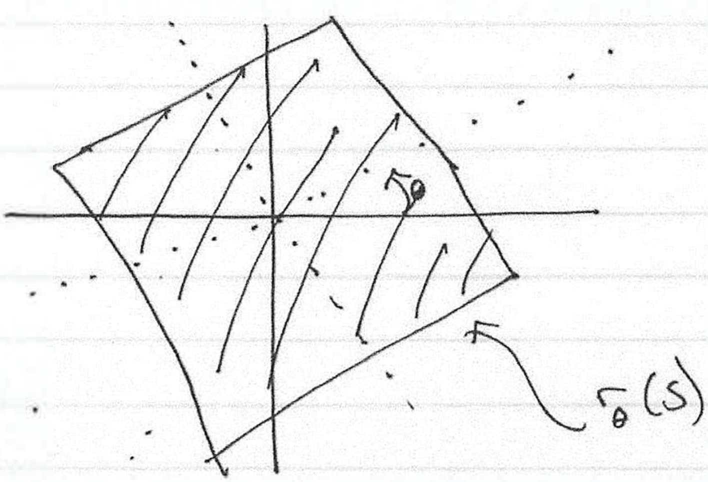
Consider rigid motions & the unit square $S$

What are the images of $S$ under rigid motions?




$$S = \{(x,y) : |x| \leq 1 \ \& \ |y| \leq 1\}$$

Image under rotations:



There are 4 rotations that "preserve" the square or leave the square "invariant":

$$r_\theta(S) = S \quad \text{for} \quad \theta = 0,\ \pi/2,\ \pi,\ 3\pi/2$$

Note $r_0 = r_{2\pi} = r_{4\pi}$ etc are the <u>same</u> <u>function</u>

$r_0((x,y)) = r_{2\pi}((x,y))$ for <u>every</u> $(x,y) \in \mathbb{R}^2$.

But $r_0((x,y)) \neq r_{\pi/2}((x,y))$ for some $(x,y)$

so these are <u>different</u> <u>functions</u>.

in fact, for all $(x,y) \neq (0,0)$

The rotations $r_0$, $r_{\pi/2}$, $r_\pi$, $r_{3\pi/2}$ have the property that they leave the square invariant.

This is because the square has symmetries.

Set $r = r_{\pi/2}$.

Then:

$$r^0 = id_{\mathbb{R}^2} = r_0$$

$r^0$ = $r$ to the zero-th power

$r_0$ = rotation by 0 radians

$$r^1 = r = r_{\pi/2}$$

$$r^2 = r_\pi$$

$$r^3 = r_{3\pi/2}.$$

So the 4 rotations that leave the square invariant are:

identity function on $\mathbb{R}^2$ - there a many different notions that people use to identify this. I often use $e$, $i$, or $id_{\mathbb{R}^2}$ inte-changeably



$$e = id_{\mathbb{R}^2}, \quad r, \quad r^2, \quad r^3.$$

<u>Note:</u>

$r^4 = e$ $\qquad$ $r^{-1} = r^3$

$r^5 = r$ $\qquad$ $r^{-2} =$ [redacted] $r^2$ $\qquad$ $r^m \circ r^n = r^{m+n}$

$r^6 = r^2$ $\qquad$ $r^{-3} = r$ $\qquad$ similar to multiplication rules

$\vdots \qquad\qquad \vdots$

So:

$$\underbrace{\{\ldots, r^{-2}, r^{-1}, r^0, r^1, r^2, \ldots\}}_{\text{many duplicates}} = \underbrace{\{e, r, r^2, r^3\}}_{\text{no duplicates}}$$

$$\underbrace{\{r^m : m \in \mathbb{Z}\}}_{\text{list with infinitely many duplicates}} = \underbrace{\{e, r, r^2, r^3\}}_{\text{list without duplicates}}.$$

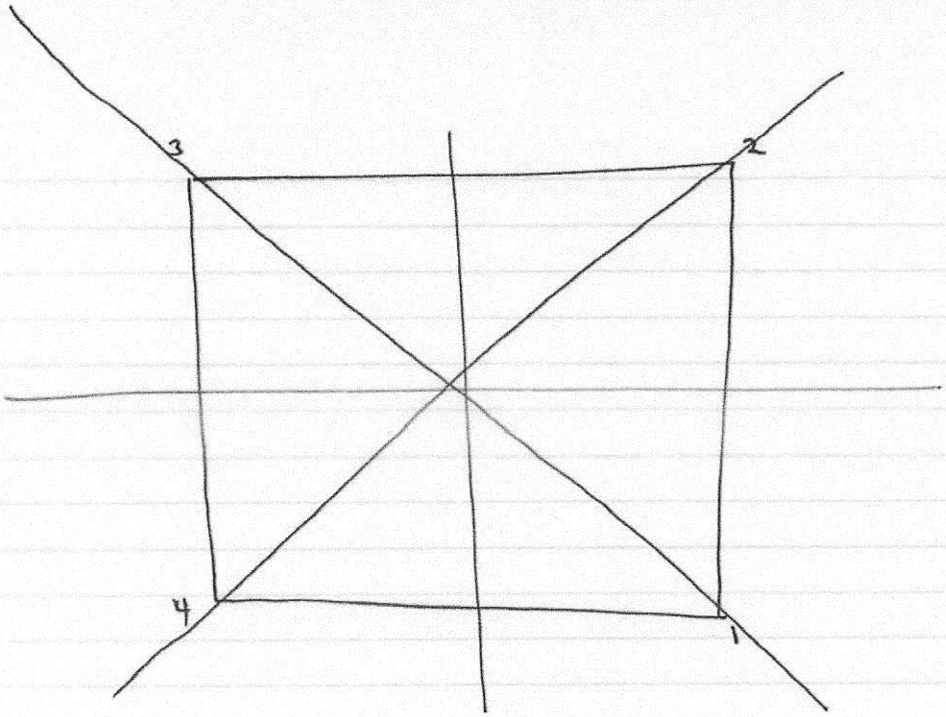
(a finite set!)

<u>Translations?</u>

<u>No</u> translations preserve the square except

$t_{(0,0)} =$ [redacted] $i_{\mathbb{R}^2} = e$ already got this.

## Flips




label corners to help you see what a flip does.

There are 4 distinct flips that preserve the square:

Call these

$a = f_0 =$ flip about x-axis $= f_{\pi}$

$c = f_{\pi/4} =$ " " diagonal $= f_{5\pi/4}$

$b = f_{\pi/2} =$ " " y-axis $= f_{3\pi/2}$

$d = f_{3\pi/4} =$ " " other diagonal $= f_{7\pi/4}$

$a, b, c, d$ are <u>functions</u>, they are the 4 flips that preserve the square.

Note $a, b, c, d \neq e, r, r^2, r^3$

these are <u>different</u> functions.

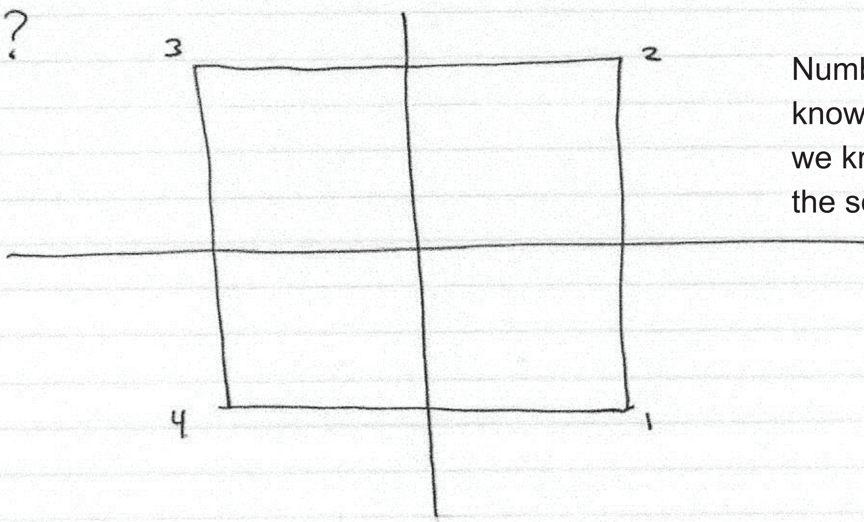
Define

$$G = \{e, r, r^2, r^3, a, b, c, d\}$$

$G$ is a set of 8 functions, each function in $G$ has the property that it preserves the square.

Any composition of 2 functions from $G$ would also preserve the square. Let's compute some of these.

$r \circ a = ?$




Number the corners. Once we know where the corners go, we know where everything in the square goes.

$r \circ a$ maps: corner $\underline{1} \longrightarrow$ corner 3

$2 \longrightarrow 2$

$3 \longrightarrow \underline{1}$

$4 \longrightarrow 4$

Same as flip c [redacted]! In fact:

$r \circ a =$ c [redacted] meaning $(r \circ a)((x,y)) =$ [redacted] c $((x,y))$ $\forall (x,y)$

No matter what 2 functions in $G$ you compose, you get another function in $G$. $G$ is <u>closed</u> under compositions.

$$e \circ r^2 = r^2$$

$$r \circ r^3 = e$$

$$a \circ b = d \qquad \begin{array}{l} 1 \to 4 \\ 2 \to 3 \\ 3 \to 2 \\ 1 \to 4 \end{array}$$

etc.

$a \circ r = d$ Note $r \circ a = c$!

Composition of Functions is not commutative in general

**Also** $G$ has an identity element:
$e \circ f = f$ & $f \circ e = f$ for all $f \in G$.

**Also** $G$ is closed under inverses.

$$\begin{array}{ll} a^{-1} = a & e^{-1} = e \\ b^{-1} = b & r^{-1} = r^3 \\ c^{-1} = c & (r^2)^{-1} = r^2 \\ d^{-1} = d & (r^3)^{-1} = r \end{array}$$

On the other hand, Composition of functions is always <u>associative</u>

**AND** Composition is associative: $a \circ (b \circ r) = (a \circ b) \circ r$ and so forth

(But not commutative!)

$$a \circ r \neq r \circ a$$

$$G = \{e, r, r^2, r^3, a, b, c, d\}$$

Thus $G$ is a set of 8 elements, together with an "operation" (composition), a way of combining elements together to get other elements, which satisfies

1. $G$ is closed under composition
2. $G$ has an identity element
3. $G$ is closed under inverses
4. The operation (composition) is associative

We proved earlier that composition of functions is always associative

We call $G$ a group because these properties are satisfied.

We can use a "multiplication table" to show all the possible compositions: 1.4

| $\circ$ | $e$ | $r$ | $r^2$ | $r^3$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $r$ | $r^2$ | | | | | |
| $r$ | $r$ | | | | $c$ | | | |
| $r^2$ | $r^2$ | | | | | | | |
| $r^3$ | | | | | | | | |
| $a$ | | $d$ | | | $e$ | | | |
| $b$ | | | | | | $e$ | | |
| $c$ | | | | | | | $e$ | |
| $d$ | | | | | | | | $e$ |

Exercise: fill in the table!

Often, we use multiplicative-like notation

Write just $ar$ instead of $a \circ r$

Call $ar$ the "product" of $a$ & $r$ (order important!)